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Johann Heinrich von Thünen
1783–1850

JOHANN HEINRICH VON THÜNEN

By Erich Schneider

Translation from German by Dr. Anne von Bibra Sutton

Johann Heinrich von Thünen was born June 24, 1783, on his father's estate Kanarienhausen in the Grand Duchy of Oldenburg, the descendant of an old feudal family. His father, Edo Christian von Thünen, who is described as a man of remarkable knowledge, especially in mathematics and mechanics, Thünen lost when he was two years old. In 1789 his mother married a merchant named von Buttel, and with the children of her first marriage moved to Hooksiel, a little town on the Jahde. Johann Heinrich von Thünen received his first instruction in the town-school of this place. We are told that he was a quiet, studious pupil, who was especially gifted in arithmetic. When he had completed his thirteenth year, von Thünen was sent to Jever to attend the secondary school. Here he occupied himself with special zeal and success with mathematics. He was always busy with some mathematical problem, and in the papers which he left from that time we find solutions of problems which give testimony of extraordinary talent.

His natural leaning towards agriculture, and the expectation of having to take over his father's estate some time in the future, induced him to devote himself to practical farming. In 1799 he went as a student to the estate of Gerrietshausen near Hooksiel, where he became thoroughly acquainted with the purely practical labors of a farm. After completing his apprenticeship, he went to the Agricultural College at Gross-Flottbeck near Hamburg, which was administered by Lukas Andreas Staudinger, who subsequently remained his lifelong, paternal friend. Here in Gross-Flottbeck, in 1803, when he was but twenty years old, he conceived the idea of the Isolated State. In a treatise, *Description of the Agriculture in the Village of Gross-Flottbeck*, he wrote:

Wenn man annähme, dass in einem Lande von 40 Meilen Durchmesser in der Mitte eine grosse Stadt läge, dass dieses Land seine Produkte nur nach dieser Stadt absetzen könnte und dass die Landwirtschaft in diesem Distrikt auf dem höchsten Grade der Kultur stände, so könnte man annehmen, dass die Wirtschaftssysteme um diese Stadt sich in 4 Klassen teilen würden.[1]

Schumacher here notes that it was clear even to this mere lad that there was no ideal of agriculture which would be applicable to all circumstances, but that the distance of a farm from its market, the prices of products, the richness of the soil, and so forth, were of decisive influence in the choice of a system of agriculture, in order to get the

[1] H. Schumacher-Zarchlin, *Johann Heinrich von Thünen, Ein Forscherleben* (Rostock, 1868), pp. 15–16.

highest possible net revenue. "Even as a wanderer in the dark forest listens to the mysterious rushing of water as yet invisible, the youth here listens gropingly at the very source of immortal laws. As yet the water gushes forth troubled and over rough stones. It was to be through the labors and cares of the man, the diligence and night watches of the scientist, that he was finally to succeed in damming its shores so that it might flow on as a clear stream."[2]

During this same year von Thünen attended the lectures of Thaer in Celle, whose work had the greatest influence on him. In Celle he was also busy with his mathematical studies. In 1803 he went to the University of Göttingen. But he abandoned his courses after two semesters. During his college year he made the acquaintance of Helene Berlin, the sister of one of his fellow-students, and became engaged to her in the autumn of 1804. Following the advice of his father-in-law, he sold his paternal estate in the Jever country, was married in the year 1806, and leased the estate of Rubkow near Anclam. A life of care and worry began. A mediocre soil produced only small harvests; in addition to this it was a time of war, there were quarterings of soldiers, and cattle-pests, and altogether he was spared none of the bitter experiences of life. June 24, 1808, was finally the day of release. He left Rubkow and went with his family to the house of his brother-in-law, and from here he quietly looked around for a new farm. In 1810, after von Thünen had confined himself to a closer choice of thirteen farms, he bought the estate of Tellow in Mecklenburg, and the soil of this farm became to him an inexhaustible source of deep scientific truths. Here, in calm and quiet work, he finally had the opportunity to win from his own estate the necessary foundations for his research studies, the data which other farm accounts had only given him spasmodically up until now.

After ten years of the most intensive labor he attained his first goal. On New Year's Eve of the year 1820 he wrote to his brother:[3]

Der heutige Tag wird in meinem Leben einen bedeutenden und angenehmen Abschnitt machen, denn ich habe heute eine 10-jährige, höchst mühsame Arbeit vollendet. Als ich vor 15 Jahren zuerst den Gesetzen über die Aussaugungskraft der Gewächse usw. auf die Spur kam, wurde ich von diesen Ideen begeistert; sie schienen mir wichtig genug, um ihrer Fortbildung mein Leben zu widmen. Es war für mich eine schöne Zeit, als ich, meiner Fantasie freien Spielraum lassend, Schlüsse auf Schlüsse baute und immer zu neuen Entdeckungen fortschritt. Aber ich bemerkte zu meinem Leidwesen bald, dass alles, was ich auf diese Weise schuf, in seinen Endresultaten doch nie mit der Wirklichkeit übereinstimmen konnte und dass, wenn ich etwas wahrhaft Nützliches und praktisch Brauchbares hervorbringen wollte, ich mir die *Grundlage zu meinem Kalkül erst*

[2] H. Schumacher-Zarchlin, *op. cit.*, pp. 15–16.

[3] All passages from the letters of von Thünen are quoted from H. Schumacher-Zarchlin, *op. cit.*

aus der Erfahrung entnehmen müsse.[4] Als ich dies klar erkannt hatte, legte ich mir das harte Gesetz auf, mit dem Fortschreiten in den Ideen inne zu halten und alle Kraft und Zeit auf die Erforschung der Wirklichkeit zu verwenden. Dies wurde nun für meine nächste Lebenszeit entscheidend. Ich fing die Tellow'schen Rechnungen in einem solchen Umfange an, als ich nur irgend ausführen konnte und als der Zweck meines Kalküls erforderte. Arbeitsrechnung, Korn-und Geldrechnung mussten gleich umfassend and gleich genau geführt werden, und dies musste fast alles von meiner Hand geschehen, weil sonst dem Ganzen Einheit und innere Glaubwürdigkeit gefehlt hätte.

Die Natur beantwortet das, was ich suche in jeder Wirtschaft und doch muss jeder, selbst der wissenschaftlich gebildete Landwirt, es erst durch eine lange and kostbare Erfahrung lernen, weil jeder die Mühe des Aufzeichnens scheut, und so jede frühere Erfahrung wieder verloren geht.

Ich ahnte freilich anfangs nicht, welche Mühe und Arbeit ich mir durch diese Rechnung auflegte. Ich habe fast die ganze Muhse der Winterzeiten dazu verwenden müssen; ich habe den geselligen und häuslichen Freuden, ja selbst dem Studium der andern Wissenschaften zum Teil entsagen müssen. Als nun die Natur selbst durch körperliches Uebelbefinden und durch Augenkrankheit mir hindernd in den Weg trat, da war ich mehrmals im Begriff, die ganze Arbeit unvollendet liegen zu lassen; aber ein innerer Drang und der feste Wille, einen einmal gewählten Lebenszweck nicht aufzugeben, verliehen mir den Mut auszuharren.

Jetzt liegt eine *zehnjährige Rechnung* vor mir; das langersehnte Ziel ist erreicht. Freilich bedarf ich nun noch der Muhse einiger Jahre um die gesammelten Data zu ordnen und für andere nützlich zu machen; aber jede hierauf verwandte Arbeit gibt ein Resultat, ist lohnend und angenehm zugleich. Jetzt werde ich die Früchte früherer Anstrengung ernten und für meine künftigen Beschäftigungen öffnet sich eine sehr frohe Zukunft.

Was ich "über den Einfluss der Dungkraft des Bodens und der Getreidepreise auf die Ackerbausysteme" gedacht und aufgefunden habe, habe ich in verschiedenen Jahren successive niedergeschrieben; aber es bedarf einer zweiten Bearbeitung und Umarbeitung, um zur Einheit zu gelangen, und um die zehnjährige Erfahrung zu Grunde zu legen. Der Wunsch, diese Schrift zur Publicität zu bringen, hat für mich fast allen Reiz verloren: sie wird von dem grössten Teil der Landwirte nicht verstanden werden, von dem andern Teil, deren Lehren dadurch entkräftet und widerlegt werden, angefeindet werden.

Wenn man das Treiben und Wesen der literarischen Welt betrachtet, wenn man sieht, wie alles Streben nach Wahrheit verkannt wird, sobald eine kleinliche persönliche Leidenschaft des Beurteilers damit kollidiert; wenn man sieht, wie die grössten Männer sich anfeinden, weil persönliche Rücksicht ihnen mehr gilt als die Wissenschaft, so muss man wahrlich alle Lust verlieren, sich in diese Sphäre zu begeben. Ungerechter Tadel würde mich kränken, übertriebenes Lob mir verderblich werden, indem ich teils noch nicht unempfindlich genug dagegen bin und hauptsächlich, indem es mich in eine Menge von Verbindungen und Besuche stürzen würde, die mit meinem Vermögen und meiner häuslichen Lage gleich unverträglich sind. Andererseits ist die Bearbeitung dieser Ansichten nun einmal Zweck meines Lebens geworden. Für das unaufhaltsame Hinstürmen der Zeit finde ich nur darin Beruhigung, dass mein Leben, wenn es einst endet, nicht spur-und nutzlos verschwunden sein mag.

[4] Italics here and in the following quotations are mine.

Es ist nun mein Plan, an der Ausarbeitung dieser Schrift nur langsam zu arbeiten, damit aber das Studium aller Wissenschaften, die in einiger Verbindung damit stehen, zu vereinigen. Durch die Beziehung, worin diese Wissenschaften mit meinem Gegenstande stehen, erhalten sie ein weit lebendigeres Interesse für mich—und diese Beziehung ist sehr weit, "in ihren Wipfeln vereinigen sich alle Wissenschaften" sagt Frau von Staël-Holstein.—Ich hoffe auf diese Weise meinen Beruf mit meiner Fortbildung zu vereinigen, und meine Muhse auf eine höchst angenehme Weise zu verwenden.

It was only after six years that the first part of the book was ready for printing under the title *The Isolated State in Relation to Agriculture and Political Economy, or Researches on the Influence which the Price of Wheat, the Richness of the Soil, and the Market, Have on Agriculture.*[5] However, Thünen could not make up his mind to publish the book because he was afraid of being attacked and misunderstood. On December 29, 1821, von Thünen wrote to his brother:

Der *ideale* Staat ist darüber nun ganz vergessen und bei meinen vielen Nebenbeschäftigungen darf ich nun nicht mehr hoffen, seine Ausarbeitung zu vollenden. Auch habe ich jetzt eine gewisse Abneigung gegen alles Bekanntmachen. Neue Ansichten in diesem Fache des Wissens werden von den wenigsten verstanden; diejenigen, die sie verstehen, haben in demselben Fache gearbeitet und sind in ihrem Urteil befangen, indem sie diese Ansichten nur nach dem Standpunkt ihres Systems auffassen und sie verwerfen oder billigen, je nachdem sie mehr oder weniger mit ihrem System übereinstimmen. Dies sollte nicht von der Bekanntmachung abhalten; *denn die Wahrheit wie der Irrtum haben immer den Kampf gegen die herrschende Meinung bestehen müssen;* aber so lange dies noch eine Abneigung erweckt, bleibt die Bekanntmachung allemal ein Opfer, und um das zu bringen, braucht man nicht zu eilen. Hast du Malthus Werk über Volksvermehrung schon gelesen? Wenn du auch, um dies Werk zu lesen, einige Collegia versäumen müsstest, so wird es Dich doch nicht gereuen.

His friends, however, succeeded in persuading him to put aside these apprehensions and to give his permission for printing. Perthes in Hamburg undertook the edition. One could not expect that a book which was so completely different from the usual way of thinking should find wide understanding and acceptance.

Dein Werk ist dieser Generation, die nur weiche Speise in Brei und Ragoutgestalt, wie der zu früh verstorbene Besser sagte, vertragen kann, zu stark und kräftig," schreibt Staudinger an Thünen.

Indeed, the reception which the book had in Germany was not encouraging for Thünen in spite of all the brilliant reviews and the manifold honors which he received. The textbooks on agriculture and political economy quoted his views, many different publications tried to secure him as collaborator. the University of Rostock bestowed upon him the Degree of Doctor of Philosophy *honoris causa:*

[5] Originally Thünen intended to give his book the title *The Ideal State.* At the instigation of his brother, the expression *ideal* was supplanted by *isolated* in 1823.

Es scheint mir fast, dass es für das Publikum kein Bedürfnis ist, über die Gegenstände, die mich von jeher am lebhaftesten interessiert haben, zur Klarheit zu gelangen. Wenigstens ist unter allen Rezensionen meiner Schrift keine einzige, so lobend sie auch sein mögen, die *in das eigentliche Wesen derselben eingegangen ist*, und durch gerechten Tadel mich gefördert und zum Weiterarbeiten gereizt hat. Dem Freund und Bruder, der mich kennt, darf ich dies wohl sagen, ohne dass er Anmassung darin finden wird.

Only after the appearance in 1842 of the second edition, which had been provided with supplements and improved in certain points, did the world begin to discover the gold which was hidden in this book. Recognition came from all sides, and Tellow became a place of pilgrimage for the most distinguished men of all countries.

Ich stehe keinen Augenblick an, seine (Thünens) Arbeit für eine der allerbedeutendsten zu erklären, welche in Deutschland für die exakte Staatswissenschaft geschehen sind. [Roscher in 1845]

Even during the printing of the first edition of *The Isolated State* in April of 1826, Thünen continued his researches on the relation of the rate of interest to the wage, which were to occupy him for the following years. On April 12, 1826, he wrote to his brother:

Es ist mir während einer Krankheit, die nur bloss körperlich schmerzhaft war, den Geist aber völlig frei liess, gelungen, über den Zinsfuss zu einer mich jetzt noch befriedigenden Klarheit zu gelangen. Dies lang ersehnte Licht ging mir in einer schlaflosen Nacht . . . vom 3. auf den 4. Februar auf. Noch weiss ich die Verbindung zwischen den gefundenen Sätzen und der Wirklichkeit nicht zu finden; aber dies muss sich wohl ergeben, wenn ich Muhse und Geistesruhe haben werde.

On March 8, 1829, he again wrote to his brother:

Im Herbst habe ich mich wieder lebhaft mit der Untersuchung über den Zinsfuss beschäftigt und am letzten Tage des Jahres gelang es mir, die Untersuchung bis zu einem Punkte zu führen, wo die Entscheidung ganz nahe liegt. Seitdem habe ich die Sache ruhen lassen und mich nur mit Rechnungen und mit anderen Nebensachen beschäftigt, aber es ist mir, als sei mein Wesen gespalten, als sei ich mit mir selbst uneins, wenn ich mich nicht einer ernsten, mich ganz beschäftigenden Untersuchung hingebe, und ich fühle dringend das Bedürfnis, bald dazu zurückzukehren.

These researches seem to have come to a preliminary conclusion with the finding of the well-known formula of the natural wage, as we see from a letter addressed to his brother on November 7, 1830:

Ich wurde mit solcher Gewalt zu meinen früheren, schon jahrelang fortgesetzten Untersuchungen über das Verhältnis zwischen Zinsfuss und Arbeitslohn zurückgetrieben, dass ich innerhalb 4 Wochen keines anderen Gedankens fähig war, obgleich meine Gesundheit sehr darunter litt. Endlich ging mir das ersehnte Licht auf, und gross war die Belohnung der Anstrengung.

He then presented to his brother some of his main results and continued:

Das hier in Worten Ausgesprochene geht aus der Untersuchung in rein mathematischen Formeln hervor, und Du kannst Dir denken, mit welchem Entzücken ich diese Formeln betrachte.

However, as long as the relation between Capital and the Product of Labor was not known, no practical use could be made of the found formula. Eighteen years more of hard labor were necessary to bring this problem also to a successful solution.

Gleich nach Neujahr hatte ich den Entschluss gefasst, alles andere aufzugeben und mich ganz der *Untersuchung über das Verhältnis zwischen Arbeitslohn und Zinsfuss* zu widmen. Zu dieser Untersuchung bin ich von jeher durch eine wahrhaft mysteriöse Macht getrieben, und ich freute mich der Fortschritte in derselben. Mitten in dieser Beschäftigung erhielt ich Fischers misswollende, herunterreissende Rezension des isolierten Staates. Wenn ich ohnehin keine besondere Neigung in mir verspürte, für das Publikum zu arbeiten, so ward diese durch eine solche Rezension, gegen welche kein Inhalt des Buches schützen kann, aufgehoben und in Abneigung verwandelt. Dessen ungeachtet blieb der Eifer für meine Arbeit unverändert. (Brief vom 20. März an seinen Bruder). Mir ist das Glück, an der Fortsetzung meiner Schrift zu arbeiten, leider noch immer versagt geblieben. Jedoch trägt mein Befinden nicht die alleinige Schuld. *Nach meiner eigentümlichen Natur kann ich nur fortbauen, wenn ich eine mathematisch sichere Grundlage habe.* Für den naturgemässen Arbeitslohn ist der Ausdruck $\sqrt{ap}$ gefunden; dies kann absolut richtig sein, aber um davon eine Anwendung in Zahlen zu machen, *muss das Gesetz wonach q und p (Kapital and Produkt) miteinander verbunden sind, gefunden und dargestellt werden.* Die Erforschung dieses Gesetzes hat mich seit 20 Jahren beschäftigt, aber da die Wirklichkeit gar keine Data dazu liefert, leider immer vergebens. Als ich im verwichenen Frühjahr an der Fortsetzung des isolierten Staates arbeitete, stiess ich wieder auf diesen Punkt, und meine Arbeit blieb liegen. Seitdem hat dies Problem mich während des ganzen Sommers täglich beschäftigt, und ich bin der Lösung entschieden näher gekommen, aber völlig befriedigend gelöst ist es immer noch nicht. Zwar sind schon aus der Kenntnis, dass der Arbeitslohn $\sqrt{ap}$ ist, für mich die wichtigsten Resultate hervorgegangen, aber soll ich mit wahrer Freude fortarbeiten, muss ich die Verbindung zwischen q und p kennen. Diese Untätigkeit ist mir umso peinlicher, da ich gerade in diesem Sommer so viele Beweise erhalten habe, dass mein früheres Streben nicht ein vergebenes gewesen ist. [Brief vom 26. November 1845 an seinen Bruder.]

In 1848 finally he could write full of joy to his son Heinrich:

Das Problem, eine Gleichung zwischen Kapital und Arbeitsprodukt zu finden, womit ich mich seit 20 Jahren vergebens abquälte, was mich nicht zur Ruhe und nicht zum Fortarbeiten kommen liess, erhellte sich mir plötzlich, und ich habe in den 3 Tagen vom 17.–19. Januar eine Skala gefunden, die allen meinen jetzigen, ob auch den künftigen weiss ich nicht, Ansprüchen genügt. Ob der Gewinn so gross ist, als das Streben danach anzudeuten schien, mag zweifelhaft sein; es hat vielleicht für keinen anderen Menschen als für mich ein Interesse, aber wie oft ich mir dies auch sagte, es half nichts, das Problem liess mir keine Ruhe und hinderte mich am Fortarbeiten. Jetzt bin ich mindestens einen Quälgeist los und mit der weh-

mütigen Erinnerung an diese drei Tage mischt sich eine erheiternde. In den Tiefen des menschlichen Geistes liegen Geheimnisse verborgen, die uns nur selten und nur durch besondere Veranlassung zum Bewusstsein gelangen.

Still Thünen could not make up his mind to publish the researches which he had made after the appearance of the first edition of *The Isolated State*, that is after 1826. It was only due to the urging of the Revisionsrat Schumacher and his son that he finally declared himself willing to publish the researches on the Natural Wage and its relation to the Rate of Interest and to Rent of Land.

Gestehen muss ich noch, dass ich mir selbst gewissermassen untreu geworden, indem ich den beiden Schumacher (Vater und Sohn) nachgegeben und das erste Heft des zweiten Teils des isolierten Staates dem Druck übergeben habe. [Brief vom 18. Januar 1850 an seine Tochter.]

He did not live to see the effect which these researches to which he had devoted twenty years of his life had on the specialists in his field. On September 22, 1850, a stroke of apoplexy put an end to his full and rich life. His friends and relatives escorted him to his last rest. In fulfillment of a wish which he had once uttered, his tombstone carries as inscription that result of his laborious researches which seemed most important to him, namely the relation of the Wage to the Rate of Interest and the Rent of Land:

$$\text{The Natural Wage} = \sqrt{ap}.$$

A thorough perusal of Thünen's rich scientific legacy was made by his friend and pupil, H. Schumacher. The most valuable manuscripts, among them especially the important researches on the relation between Capital and the Product of Labor and the numerical bases for the investigations in *The Isolated State*, were published by Schumacher in the year 1863 as the second part of the second volume of *The Isolated State*. A third volume published by Schumacher in 1863 contains the "Grundsätze zur Bestimmung der Bodenrente, der vorteilhaftesten Umtriebszeit und des Werts der Holzbestände von verschiedenem Alter für Kieferwaldungen."

II

For a long time most of the scientific world looked upon Thünen's work as a *noli me tangere*, which was admired with reverend awe by most and understood by few. Thünen, with Gossen and Cournot, shared the fate of all great men who leave the travelled paths of traditional thought and work, and thus, hurrying far in advance of their own time, become the pathfinders for a later scientific generation. Even today we must sadly admit that Thünen's unique accomplishments still stand on a lonely pinnacle far from the great broad highway which

leads through the realm of our science. Only few dared, and still dare today, to make the difficult ascent to this summit; the greater number of those who started out to conquer it stopped halfway, and therefore still try to find his importance in secondary things and in other directions from where it really lies. It is certainly not wrong, as Ehrenberg did, to see in him a representative of detailed research in the field of economics, or to claim him as an empiricist, or to praise him as a great social politician. But that is not going to the core of his labors, it does not characterize his true scientific being. What he really was has been strikingly expressed by Marshall who, as he himself gratefully recognizes, was lastingly influenced by him. "Von Thünen was a bonâ fide mathematician, but of less power: his blunder as to the natural wage is not of the same order as Cournot's little slips. But, to make up, *he was a careful experimenter and student of facts, and with the mind at least as fully developed on the inductive as on the deductive side.* Above all he was an ardent philanthropist . . . I loved Thünen above all my other masters."[6]

These qualities and abilities which Marshall praises in Thünen, are the very ones which mark an economist as an econometrist. Thünen was one of us, one of the great pioneers on the way to a goal which is beginning to become universal. He did not descend from the height of his ideas down to reality, but he ascended from reality to ideas. He shows us with classic examples "how theory grows out of the observation of business practice."[7] Only he who sees in Thünen the great econometrist sees him correctly, and has the proper approach to his work. Thünen may have made mistakes in individual results, as for instance in the deduction of the famous formula for the Natural Wage, which he himself considered to be the crown of his achievements, to which instance critics, by the way, confined themselves almost exclusively, and herewith believed they had found again a case by which could be demonstrated in a concrete and convincing manner the absolute inefficacy of mathematical reasoning for the solution of economic problems, instead of seeking the mistake in the assumptions and in the reasoning as a whole. However, this changes nothing of the fundamental importance of his mission for the development of econometrics; and does not in the least minimize its enduring value.

Of course he alone really understands the true content of the message which Thünen's work carries who treads the thorny path that leads him to the very end. Whoever does not dread this labor will be richly

[6] *Memorials of Alfred Marshall* (Edited by A. C. Pigou), London 1925, p. 360. Italics are mine.

[7] Schumpeter in ECONOMETRICA I, p. 9.

rewarded. He will recognize Thünen as the master of theoretical methods of work, and find that ideas which today we use as a matter of course in our daily work, have been formed and executed by him with rare clearness. Let us only sketch shortly the most essential points and verify them with the corresponding quotations from *The Isolated State*.

1. Thünen saw with remarkable definiteness that the economic mechanism is a great system of functional relations.

Während man in den meisten Wissenschaften die Untersuchung mit einzelnen feststehenden und als gegeben betrachteten Sätzen beginnt, haben wir es hier mit Potenzen (d. h. Grössen, d. V.) zu tun, die in einer steten Wechselbeziehung zu einander stehen, und wovon keine einzige als gegeben angenommen werden darf.

Dadurch aber wird unsere Untersuchung so schwierig und verwickelt—und es fragt sich, ob so viele Gleichungen gefunden werden können als zur Bestimmung der unbekannten Grössen erforderlich sind. [*Der isolierte Staat*, 2. Teil, 1. Abt., p. 532.][8]

2. Thünen convincingly proves that only the method of successive approximations is suitable for the disentanglement of the variegated tangle of economic interrelations.

Ich bitte den Leser . . . sich durch die . . . von der Wirklichkeit abweichenden Voraussetzungen nicht abschrecken zu lassen und diese nicht für willkürlich und zwecklos zu halten. Diese Voraussetzungen sind vielmehr notwendig, um die Einwirkung einer bestimmten Potenz—von der wir in der Wirklichkeit nur ein unklares Bild erhalten, weil sie daselbst stets im Konflikt mit andern gleichzeitig wirkenden Potenzen erscheint—für sich darzustellen und zum Erkennen zu bringen.

Diese Form der Anschauung hat mir im Leben über so viele Punkte Licht und Klarheit gegeben und scheint mir einer so ausgedehnten Anwendung fähig, dass ich sie für das wichtigste in dieser ganzen Schrift halte." [Aus dem Vorwort zur 2. Aufl. des 1. Teiles des *Isolierten Staates*.]

3. The idea of the static equilibrium state and the importance of the construction of this term for economic theory, is clearly understood by Thünen, and he develops it in a way which deserves our fullest admiration.

"In dem isolierten Staat ist der verharrende Zustand Grundlage der Betrachtung" (p. 419) "Wir haben stets den endlichen Erfolg, also das erreichte Ziel, vor Augen gehabt. Mit dem erreichten Ziel tritt Ruhe und damit der beharrende Zustand ein—und hier erblicken wir Gesetzmässigkeit, während in der Uebergangsperiode manches uns als ein unentwirrbares Chaos erscheint. Der beharrende Zustand kann aber aus folgenden Gründen in der Wirklichkeit nicht stattfinden." Thünen entwickelt jetzt diese Gründe und fährt dann fort: "Aber

[8] All quotations, unless marked otherwise, refer to Waentig's edition of *The Isolated State* in the *Sammlung sozialwissenschaftlicher Meister*, (Jena: Fischer), vol. XIII.

trotz dieser Wandelbarkeit liegt in dem einzelnen, was wir der Betrachtung unterziehen der Keim zu einer bestimmten . . . Entwicklung, und wie wir wissen, welcher Baum aus der in die Erde gelegten Eichel einst hervorgehen wird, so können wir auch hier die aus der Entwicklung des Keims entsprossende Frucht—den endlichen Erfolg—unter der Voraussetzung, dass keine störenden Einwirkungen stattfinden, im Voraus erkennen und im Geiste anschauen. Hierin aber liegt die Berechtigung, bei unseren Untersuchungen den beharrenden Zustand ins Auge zu fassen und zu Grunde zu legen.

Die durch diese Methode erlangte Erkenntnis kann aber wesentlich dazu beitragen, über die verwirrenden Erscheinungen während der Entwicklung und des Uebergangs Lichts zu verbreiten. (p. 432)

4. But especially—and in this we may see his greatest achievement from the point of view of the pure theorist—is Thünen the ingenious creator of the instrument of marginal analysis, which he employed with such admirable skill and which led him to results that even today belong to the fundamentals of modern economic theory. The application of marginal analysis for the determination of the amount of wages and the amount of the rate of interest under conditions of competition then leads him to the fundamental problem of the Imputation theory.

Das Arbeitsprodukt ist das gemeinschaftliche Erzeugnis von Arbeit und Kapital.

Wie ist hier nun der Anteil, den diese beiden Faktoren jeder für sich an dem gemeinschaftlichen Produkt haben, zu ermessen?

Die Wirksamkeit des Kapitals haben wir ermessen an dem Zuwachs, den das Arbeitsprodukt eines Mannes durch die Vergrösserung des Kapitals, womit er arbeitet, erlangt. Hier ist die Arbeit eine konstante, das Kapital aber eine veränderliche Grösse.

Wenn wir dieses Verfahren beibehalten, aber umgekehrt das Kapital als gleichbleibend, die Arbeiterzahl als wachsend betrachten, so muss auch bei einem Betrieb im grossen die Wirksamkeit der Arbeit durch den Zuwachs, den das Gesamtprodukt durch die Vermehrung der Arbeiter um einen erhält, der Anteil des Arbeiters an dem Produkt zu unserer Kenntnis gelangen. (p. 584)

5. It is evident that a scientist like Thünen, who stresses the quantitative character of economic theory with such emphasis, positively takes his stand on the side of the "mathematical method" and lays stress upon the use of mathematical methods for the investigation of the complex phenomena of agriculture and economics.

Die Anwendung der Mathematik muss doch da erlaubt werden, wo die Wahrheit ohne sie nicht gefunden werden kann.

Hätte man in anderen Fächern des Wissens gegen den mathematischen Kalkül eine solche Abneigung gehabt, wie in der Landwirtschaft und der Nationalökonomie, so wären wir jetzt noch in völliger Unwissenheit über die Gesetze des Himmels; und die Schiffahrt, die durch die Erweiterung der Himmelskunde jetzt alle Weltteile miteinander verbindet, würde sich noch auf blosse Küstenfahrt beschränken. (p. 569)

Up to this point, that which Thünen has given us has today become general knowledge, and if this had been the sum total of his achievements, it would now be only of interest from the point of view of the history of our science. But even if the historian may see in Thünen merely the great theorist of a past time, his work means more to the present student of pure theory than just a climax in the history of economic theory. Only a few of us realize that problems which now stand in the very center of our present investigations had been already recognized by von Thünen in their full significance with incomparable clearness; and that, moreover, there are contained in his books many valuable hints and data as to their efficient treatment. One needs but read carefully the introduction of the first part of the second volume of *The Isolated State* to be convinced of this; and one will be surprised to find problems posed in it, which belong to our own day and which even now await a satisfactory solution:

Wie wirkt die Grösse des Geldstocks auf den Zinsfuss und auf den Preis der Waren? (p. 402)

Welchen Einfluss üben bedeutende Verbesserungen im Landbau und Erfindung neuer Maschinen für die Fabriken bei ihrem ersten Auftreten aus, und welches ist die endliche Wirkung derselben? (p. 402)

Wie muss mit der grösseren oder geringeren Entfernung von der Handelsstadt sich die Form der Wirtschaft ändern, wenn der Boden den höchsten Reinertrag geben soll? (p. 404)

Durch welches Gesetz wird die Landrente bestimmt, wenn statt der einen grossen Stadt lauter kleine Städte von gleicher Grösse und in gleicher Entfernung voneinander in der Ebene des isolierten Staats zerstreut liegen, und in welcher Verbindung steht hier der Grad der Sorgfalt der Arbeit mit den Kornpreisen? (p. 423)

Wie kann unter gegebenen, ganz bestimmten Verhältnissen ermittelt werden, welche Grösse die Güter haben müssen, damit der Boden die höchste Rente gibt? (p. 428)

Hat die grössere oder geringere Entfernung vom Markplatz einen Einfluss auf die zweckmässigste Grösse der Güter? (p. 428)

Welchen Einfluss hat das Steigen des Bodenreichtums auf die zweckmässigste Grösse der Güter? (p. 428)

This is only a very small part of the great wealth of suggestions and fruitful thought which have been given to us here. For the econometrist, Thünen's work is moreover a source of most valuable, as yet entirely unused, and almost entirely unnoticed, material for econometric researches. I only refer to the statistical investigations on the Relation between Capital and the Product of Labor in the second part of the second volume of *The Isolated State*. In this respect it has an incalculable value.

"There will be a time when one will eagerly bring to the light of day

the pure precious metal from its rich mines," Staudinger wrote to Thünen in 1826. Today, about a hundred years after the first appearance of the book, we have finally progressed far enough to understand and appreciate the great wealth of this work of genius. Now is the time to bring to light the treasures which lie hidden in this immortal book, and to complete that which its creator had to leave unfinished.

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ANNUAL SURVEY OF SIGNIFICANT DEVELOPMENTS IN GENERAL ECONOMIC THEORY

By J. TINBERGEN

THIS survey may be opened with some apologetic remarks. The first is that to write a complete survey would mean to write a book. This has not been the intention of the editor, nor does it lie within the power of the present writer. Therefore, this survey is, in principle, incomplete, as will be easily seen by the many important authors whose work is not here discussed. The second remark is that a survey like this is, of necessity, one-sided and restricted. It is one-sided by reason of the special view of the writer on what is interesting and what is not; as, for example, this survey shows special interest in quantitative problems. It is restricted as a consequence of the fact that knowledge of nearly every individual about subjects more removed is less than about subjects in his direct neighborhood—both in a concrete and an abstract sense. The third remark relates to the subdivision of the material; it is difficult to represent, in a one-dimensional exposition, as the current text of a paper, a thing of multi-dimensional character.

First, we may consider the development of static theory in the last few years.

I. STATIC THEORY

In this field the Walrasian theory of market price may be taken as our starting point. This theory may be formulated in the following way: (1) To start with, it is assumed that a large number of subjects are present both on the "demand side" and on the "supply side" of the market. Each of these subjects realizes that he by himself will have little influence on price; he considers price as a given quantity. The only choice he can make consists in choosing the amount of money he will spend or the quantity of goods he will offer for sale per unit of time. (2) With the price given he will choose such an amount or quantity as will maximize his utility or ophelimity function, which is a given function of the quantities of different goods he buys or retains from production. (3) After this step the total amounts demanded and offered are confronted; the equilibrium price must be such as to equalize total demand and total offer, which means that we are confronted with the problem of the uniqueness of the solution.

Theoretical development in the last few years may appropriately be described by a separate consideration of each of these three elements in the argument.

The first point then is what we might call the *organization principle* of the market, culminating in the assumption that price is considered

as a constant by each subject in the market. As was pointed out by many authors, this assumption is only plausible when the number of subjects is very large. Now in modern times there are numerous markets where only a few number of sellers or of buyers appear. The simplest case is that of a monopolist, and the theory of monopoly was developed in a brilliant way many years ago. The most difficult cases are, however, those called *restricted competition* or polypoly, where the number of sellers is small, but not "one." This case was considered particularly by Cournot, as early as 1838; later on, some discussion took place as to the validity of his solution, but in general little attention has been paid to this question. Especially in the last decade an intensive interest in these questions has developed, and the newer theoretical literature has made particular advances in this respect.

The difficulties in this problem arise from the fact that as soon as sellers[1] are supposed to be able to influence prices, their ophelimity (or, more especially, their profit, to take a more realistic case) depends on quantity sold as well as on price. And this quantity is not, as in the case of monopoly, determined by price. Thus each seller seems to possess two "degrees of freedom," i.e., two independent ways of increasing his profits, and each way seems to lead to a "maximum condition." The number of equations, then, surpasses the number of variables, and no solution is to be found, in general. In other words, although each seller has, indeed, two ways to increase his profits, his action in certain respects hinders his colleagues, and is hindered by what his colleagues do; the question is in what way those interactions should be accounted for and in what way the case can be "styled" to get an appropriate picture of reality.

In Cournot's solution it was assumed that (1) each seller managed his quantity only, and (2) that he supposed the quantities offered by his colleagues not to change; further on it was assumed, in accordance with the Walrasian market, that (3) only one price exists at one time and that the total demand is determined by this price.

Those assumptions granted, Cournot's solution is, without doubt, the right one, as was pointed out anew by Wicksell and Schneider.[2] Another question is, however, whether these assumptions are the only possible ones; what other hypotheses might be chosen; and which are fulfilled in each concrete case. Zeuthen showed that assumption (2) might be generalized, Hotelling gave alternatives as to (1), bringing

[1] The argument may be used for buyers as well.

[2] Wicksell, "Mathematische Nationalökonomie," *Archiv für Sozialwissenschaft und Sozialpolitik*, LVIII (1927), 252; Amoroso, "La Curva Statica di Offerta," *Giornale degli Economisti*, XLV (1930), 1; and Schneider, *Reine Theorie Monopolistischer Wirtschaftsformen* (Tübingen, 1932), p. 132 seq.

about changes in (2) and (3), and Chamberlin gave the more general view on the problem showing that there is indeed room for different assumptions.[3] Zeuthen's[4] contribution, given in a geometrical way, and generalized by Frisch,[5] consisted in the treatment of several cases in which it was supposed that sellers take into account *certain reactions of the other sellers* for the case they themselves would change their variables.

Hotelling[6] considered the case in which sellers manage prices instead of quantities. As soon as this is supposed, however, assumption (3) is impossible, and a refinement in this respect of the Walrasian market theory is the consequence: it must be assumed, in addition, that buyers have, in certain circumstances, reasons to buy at a higher price than is necessary. They must have a reason to choose, in certain circumstances, from two sellers asking different prices, the one with the higher price. Therefore, in addition to the above mentioned reactions of other sellers, there have to be considered *reactions of buyers*, the nature of which must also be given; the number of necessary data to solve the problem increases once more.

As a concrete example, Hotelling treats the case in which geographical distance causes some buyers to prefer a higher price to a lower one; adding that other similar causes, as personal sympathy or inertia, may be treated in an analogous way. His results are in many respects interesting and seem to offer wide perspectives for the introduction of geographical and similar elements into more complex problems. Considering two sellers and a distribution of buyers grouped with uniform density along one line, and assuming that transportation costs are proportional to distance, that production costs per unit are constant and equal, that demand is absolutely inelastic, and that both sellers fix their prices supposing the other to keep his constant, Hotelling reached the following conclusions: there will be a stable equilibrium with, in general, different prices for each of the sellers; these prices will equal production costs as soon as transportation costs vanish; when one of the sellers has to choose his "Standort" he will choose one as near to his competitor as possible; this choice is contradictory to the interest

[3] Cf. also Waffenschmidt, "Cournotsche Gleichungen für beschränkten Wettbewerb," *Archiv für Sozialwissenschaft und Sozialpolitik*, LXVII (1932), 513; and Sting, "Die polypolitische Preisbildung," *Jahrbücher für Nationalökonomie und Statistik*, CXXXIV (1931), 761.

[4] *Problems of Monopoly and Economic Warfare*, London, 1930.

[5] At the Paris meeting of the Econometric Society, 1932, and in lectures at the Institut Poincaré, Paris, 1933. A brief account is published in the paper "Monopole-Polypole, La Notion de Force dans l'Économie Politique," *Nationaløkonomisk Tidsskrift*, 1933.

[6] "Stability in Competition," *Economic Journal*, XXXIX (1929), 41.

of the community, as an unnecessary amount of transportation labor has to be done.

Chamberlin[7] very clearly points out that there are two alternative possibilities as to the reactions of sellers upon the behavior of other sellers, none being *a priori* the "right" one or the "wrong" one. The two possibilities are indicated as "price adjustment" (with constancy of output) and "output adjustment" (with constancy of price). There might be room for mixed adjustment, as in Zeuthen's and Hotelling's case, or the general cases which Frisch calls "action paramétrique," but these possibilities are not considered by Chamberlin in full. He only treats one further case which represents a special case of Zeuthen's, the case, as he formulates it, in which sellers not only take into account the direct influence of their policy upon their colleagues, but also the indirect. The case Chamberlin considers consists in the extreme assumption that sellers foresee such reactions perfectly correctly and this leads to the result that equilibrium coincides with monopoly price and output (in a market with a uniform price) also when no open or tacit agreements exist. He then indicates certain causes counteracting this tendency and summarizes: "unless the number of sellers is very large, it would seem that the consumer can rely for protection against monopoly prices only upon (a) the sellers' lack of intelligence, or (b) their having only a temporary stake in the market, or (c) their uncertainty as to what their rivals will do. It is possible that these are adequate. At any rate, they are not the safeguards ordinarily contemplated by economic theory in explaining the price system."

It seems to me a real advance in our understanding of problems concerning markets with few sellers that we now realize that for a solution of those problems, *there ought to be known more than the buyers' demand and the sellers' profit functions*, and, in addition, that there may be *several types of markets*, or of sellers, according to the "reactions" above mentioned. It seems to be a further task for economists to determine the types to which real markets, or sellers, belong. It is possible, for instance, that in some markets sellers can only make price adjustments and no output adjustments. In retail trade for instance, where the ordinary "division of labor" between sellers and buyers may be formulated: sellers fix prices and buyers fix quantities. On the other hand, markets where other forms of contracts are customary, may belong to the type of "quantity adjustment markets." In very simple cases a statistical examination of the behavior of prices with the intention of finding out the type of market may be possible; in general,

[7] "Duopoly: Value where Sellers are Few," *Quarterly Journal of Economics*, November 1929, p. 63.

however, a large amount of data necessary for such investigations is still lacking.

In several cases treated by Zeuthen and Hotelling, there appear interesting problems, besides those relating to the exact place of the equilibrium point, concerning the question whether this equilibrium point will always be reached. As an example we may consider Cournot's original problem and its solution. This may appropriately be illustrated by the graphical representation given in Figure 1.[8] It is supposed there are two sellers only, the first selling the quantity x_1, the second x_2. Buyers are many and their aggregate demand function may be written as

$$p=f(x_1+x_2). \tag{1}$$

Taking as coordinates x_1 and x_2, each point in our chart indicates one market position; x_1 and x_2 can be read off directly and p is calculated

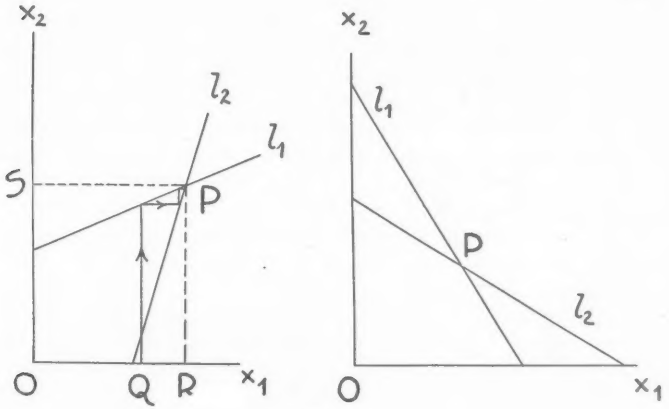

FIG. 1

with the help of (1). For each given x_2 it is possible to calculate the optimal x_1 for seller Number 1; the locus of these points is l_1; by analogy we have a locus l_2 for optimum situations for seller Number 2 when x_1 is given. The intersection point P of l_1 and l_2 gives Cournot's equilibrium point; this may be demonstrated by studying the successive movements the sellers will carry out to improve their situation—provided that they suppose their competitor's output to be held con-

[8] Given for similar purposes by Hotelling, *loc. cit.*, for the case where prices are independent variables.

stant during each "act." The latter assumption—which is not, in its limited form, essential for our argument—means that each "act of improvement" undertaken by 1 consists in a vertical movement until l_1 is reached, and each "act of improvement" undertaken by 2 consists in a horizontal movement until l_2 is reached. Starting from a situation Q, e.g., such movements will lead to P where they will stop, and in this respect P is really the equilibrium point. This holds true for a large field of different starting points Q. Under certain conditions, however, it seems not to hold true for those Q's of which either x_1 or x_2 surpasses P's x_1 or x_2, i.e., for Q's outside the rectangle $ORPS$. To explain this, a certain mechanism of the "acts of improvement" must be postulated which seems, however, not unnatural. These acts may be looked at as consisting of two parts: (1) the active seller changes his price, and attracts thereby some new demand; (2) the passive seller follows with his price and herewith prevents "his" demand from being taken off. When the acts are "constructed" in this way, only such are possible where the price change is a price decrease; as the necessity for the competitor to follow does not exist in the case of a price increase. In our graphic representation this means that vertical movements from a higher to a lower point, as well as horizontal movements to the left, are excluded. As soon as the starting point is situated outside the rectangle, P could not be reached. This statement has special importance for cases where (cf. the right side of Figure 1) l_1 and l_2 have a negative slope, as occurs in several important simple cases.

Up to this point we have discussed only problems related to a market where there is still a large number of competitors at one side. Problems grow still more difficult when at both sides a small number of competitors appears. Of those cases, only the case of monopoly at both sides has, as far as I know, been discussed rather thoroughly. I do not see, however, that much has been gained in this respect in the last few years. The most remarkable attempt to attack the difficulties seems to me to be Zeuthen's book, *Problems of Monopoly and Economic Warfare* (London, 1930), in which he tries to solve the problem in the special shape of the problem of collective bargaining. I must admit not to be sure to have understood the author's argument; it may be mentioned, however, as an original attempt, in which data are needed to solve the problem similar to the supplementary data we have found necessary to solve the problem of restricted competition at one side of the market.

Having discussed several problems relating to separate markets, we shall consider some questions that arise when a *system of markets* is studied. A first question in this field is that dealt with once more by Schneider in his book *Reine Theorie monopolistischer Wirtschaftsfor-*

men[9]; the question whether there exists, in a society in which all branches of production, including all factors of production, are monopolized, a state of equilibrium, and how it is to be determined. In this part of his book, Schneider develops a method which seems to have interesting features, although, in my opinion, he neglects to indicate the limits of its applicability. For each market considered he assumes that buyers accept—just as in the case of a unilateral monopoly—the price level as something they cannot influence, while sellers manage this price level in their own interest, i.e., in the way of the monopolist. Proceeding thus, it is possible indeed to solve this problem in an unambiguous way, and many of the criticisms on Edgeworth and Pareto added by the author seem legitimate. His assumptions on the structure of the considered markets will not hold, however, when the number of buyers is small, as may occur in different cases. Here either Cournot's mechanism or one of the other polypolistic theories must be applied—and these do not, so far as I can see, lead in all cases to a definite equilibrium point. To say it in a more popular way, a large trust that has a monopoly for its products, but, for instance, is itself one of three buyers for one of its materials, will certainly try to influence prices of this material and will, in all probability, not accept every price that is dictated.

Nevertheless, the method followed by Schneider seems to me to be very useful for further investigations in this field, a field that could be called the object *par excellence* of economics, as it compares different possible forms of organization of production. As far as I know, many interesting problems belonging to this object, and easily accessible to analysis, have not been treated yet, for example, the influence of vertical integration of horizontal trusts on prices of final products and total profits. The latter will, of course, increase, and hence one would expect prices to rise; but there are important cases in which this is not so and, on the contrary, prices fall.

Another interesting problem of interaction between markets is to be found in Von Stackelberg's *Grundlagen einer reinen Kostentheorie* (Vienna, 1932). Suppose two enterprises form a vertical amalgamation, and enterprise Number 1 supplies enterprise Number 2 with a certain commodity. Suppose, in addition, that each of the two enterprises has its own administration and price calculation, what then is the price which enterprise Number 1 should ask in order that the total profits of both enterprises are optimal?

Stackelberg's conclusion is that a distinction must be made between the case (1), where there is no market for the commodity in question,

[9] Tübingen, 1932; p. 83 ff.

and (2), where there is a market. In the first case the price should be equal to production costs, in the second case it should equal market price—or may differ from that price by an amount, inversely proportional to the quantity bought by enterprise Number 2.

Although of a quite different character, a problem of high practical importance suggested by Bauer,[10] belongs to this class of problem, which might perhaps be called "problems in the structure of modern economic life." It is the divergency that arises in respect to the effects of rationalization between private and general interest as soon as a society is considered in which an unemployment dole exists. A private enterprise bases its decisions concerning rationalization on a calculation comparing (a) income and outlay in the existing situation with (b) probable income and outlay after a certain act of rationalization. It carries through this act of rationalization as soon as (b) shows a more favorable balance than (a). The interval in which this is the case does not coincide with the interval for which this rationalization is in the interest of the whole community. Apart from the question whether a simple calculation in terms of money is sufficient to point this out, the balance of the community shows one item in addition, the doles to be paid for those losing their occupation by the act of rationalization. As soon as these payments are included in the structure of the productive community this difference occurs, and there are reasons to assume that the difference is not unimportant.

Bauer also lays stress on other causes, for as he calls it "failing rationalization," e.g., a systematic error in the estimates for (b) because of a favorable business cycle phase, ètc., but these have been recognized by earlier writers also (cf. p. 30, *Theory of Risk*). The interesting feature in the first-mentioned form of "failing rationalization" is its being connected with the new type of social measure that does not exist in the purer forms of capitalism. A further interesting question arises when the consequence is drawn from Bauer's argument that such failing rationalization should be prevented. This does not mean, as it would seem at first sight, a threatening of industrial progress, for the types of rationalization prevented would be only those giving a slight progress, while the more efficient types would be permitted. Moreover, a prevention of some forms of rationalization also seems useful in other respects. Suppose in a certain branch of industry five inventions occur, with time intervals shorter than half the life time of machinery used; then considering industry as a whole over the whole time interval, it will be more efficient to apply, e.g., inventions Numbers 1, 3, and 5,

[10] *Kapitalismus und Sozialismus nach dem Kriege*, I. "Rationalisierung—Fehlrationalisierung," Vienna, 1931.

than to apply them all in succession, as soon as investments necessary at each new invention surpass a certain limit.

With these remarks our exposition of the "organization principle" of the market may be terminated; and the second step in the Walrasian market theory, i.e., the *formation of demand and supply* may be taken as a central point for further discussion of newer literature.

In the field of (static) demand and supply functions, most progress seems to have been made in statistical directions; it is not my task to deal with this subject here. In theoretical directions an important general tendency appears to be to make a distinction between the purely "psychical" demand and supply functions and the more "technical" ones. The distinction is not quite sharp, but the central thought is important. Psychical demand and supply functions are related immediately with psychical appreciation of the goods or services in question; the most important psychical demand is the demand by consumers for consumption goods; the most important psychical offer is the offer of labor and, in some circumstances, of capital. They form, so to say, the periphery of the economic system; the inner part of which has to deal only with "technical" demand and supply functions. The analysis of the latter requires a thorough knowledge of the technical structure of modern enterprise, and this field has constituted itself a more or less independent science. Many interesting problems dealt with in the last few years relate to this field, e.g., all questions connected with overhead cost, such as dumping, discriminating price policy, etc.

The principal general economic consequences of the intensified study of enterprise economics may be seen in the recognition that there are different types of supply and demand curves relating to the same enterprise, according to the number of production factors which are kept constant.[11] This number not only depends on the estimated duration of price changes (the difference between long-run and short-run variations), but also on the intensity of such changes; a small change in output may be performed with a constant number of employees but with a varying number of laborers, whereas a larger change will be possible only by an increase in both categories. One has the impression, however, that a more general treatment of those problems has a limited significance; it will not be discussed here further.

Among the more remarkable contributions in the market-place aspect of demand and supply, two may be mentioned here. In the first

[11] Cf. e.g., Viner, "Cost Curves and Supply Curves," *Zeitschrift für Nationalökonomie*, III (1931), 23; Schneider, "Statische Kostengesetze," *Nationalekonomisk Tidsskrift*, LXX (1932) 393; Von Stackelberg, *Grundlagen einer reinen Kostentheorie*, Wien, 1932.

place various publications by Ricci[12] give a more elaborate classification of demand (or supply) curves into five or even seven groups, including different combinations of the elementary types: (1) elastic, (2) inelastic, (3) elastic-inelastic, (4) inelastic-elastic, (5) anelastic. Group (3) represents curves consisting of two successive parts, the first part being elastic, the second inelastic; group (5) is formed by curves whose elasticity is equal to one in all points. Further groups may be formed—and are formed by the author—by considering curves whose elasticity changes twice, etc. In his expositions, Ricci operates with the notion of "(marginal) utility rectangles," being the product of abscissas and ordinates for each point of the curve, which is the same notion as Schneider's "marginal outlay" or "marginal receipts." It is clear that as the concept of elasticity can be applied to every curve these groups may be distinguished as well for other functions. The discussion between Frisch and Ricci concerning the group to which the utility curve for income belongs will be mentioned later on.

In the second place an attempt by Roy[13] may be mentioned to show, with the help of a simplified example, in what way the distribution of income influences the aggregate demand for consumption goods. To fix the ideas, Roy assumes that commodities can be ranged according to their urgency, and that each individual first buys a constant quantity of commodity Number 1, or as much less as his income necessitates; of the residue, if any, he buys a constant quantity of commodity Number 2, as far as possible, etc. The influence of price on the quantity demanded is then to be ascribed to the fact that a definite price increment prevents a certain group of people from buying their "full" quantity. Although in most actual cases this seems not to be the only, and not even the most important reason for decreases of demand owing to price increases, Roy's case throws some light on the relations between division of incomes and slope of the demand curve.

A daring attempt in the last decade to make concrete the theory of marginal utility is that of Fisher[14] and Frisch, the results of which have been published together in Frisch's *New Methods of Measuring Marginal Utility*. The statistical part of these investigations will not be discussed here; for our subject, however, the theoretical assumptions that form the starting point are very interesting. Frisch starts from the equation

[12] Cf. e.g., "Die Kurve des Geldnutzens und die Theorie des Sparens," *Zeitschrift für Nationalökonomie*, III (1932), 307.

[13] "La demande dans ses rapports avec la répartition des revenus," *Metron*, 1930.

[14] "A Statistical Method for Measuring Marginal Utility, etc." *Economic Essays in Honor of John Bates Clark*.

$$p = \frac{u}{v}$$

in which p means the price spent for a certain commodity, u the marginal utility of that commodity to the buyer, and v the marginal utility of money (or income) to that buyer. There will be very little difference of opinion among economists as to the validity of this equation; it is, practically speaking, generally accepted. The next step consists in the assumption that *u depends only on the quantity of the commodity in question consumed per unit of time by the subject in question*, and that v may be considered as a function of income. Granted this assumption, very interesting conclusions are possible by comparing two situations in which the individual (or family) is consuming equal quantities, but in which income and price are different, and differ in such a particular way that the quantity consumed of the special commodity of comparison (the reference commodity) is unchanged. By writing down the above equation for each of these situations and dividing the first by the second, we find, as the two u's are now equal:

$$\frac{p_1}{p_2} = \frac{v_2}{v_1} = \frac{v(i_2)}{v(i_1)}$$

That means that the *quotient of the marginal utilities* corresponding with the two income situations *can be calculated* from the two prices. After a correction for possible changes in the general index of cost of living, which we may not enter into now, it is possible to find in this way in what proportion the marginal utility of money falls off when income rises, figures that are of great interest for all sorts of social problems.

From a theoretical point of view, two questions are especially interesting, which I may only mention without giving a definite solution for them. The first is in what degree the assumption is admissible that u depends only on the quantity consumed of this particular commodity.[15] Everyone agrees that this does not hold for commodities having a substitute; the authors, however, exclude such commodities or think them combined; so the only question remains, are there really "independent" commodities? Is the increase of utility experienced with an increase in sugar consumption from 100g to 200g per week equal for a person who in other respects lives well and for one having no roof over his head? This example is, of course, an extreme one. As a first approximation, in cases which do not diverge so widely, it seems to me very

[15] At the University Institute of Economics, Oslo, work is actually going on to investigate this aspect of the problem.—EDITOR.

appropriate, however, to assume that there exist commodities (or services) that may be considered independent for the present purpose.

Another interesting question seems to me to be in what degree it is permitted to compare different individuals. By taking averages for large groups, individual differences may be eliminated. There is also the question of habit. Will a group having earned for a long time $2000, and now earning $3000, behave identically with a group always having earned $3000? Again, as a first approximation, I consider the assumption of identity as appropriate, but the possibilities referred to must be borne in mind.

A controversy between Ricci[16] and Frisch[17] relating to change in money flexibility may be briefly mentioned here. It concerns the question as to which of the types of curves distinguished by Ricci (see above) the utility curve of money belongs. According to Frisch it is of the inelastic-elastic type, according to Ricci it is of the elastic-inelastic type. Frisch's opinion includes two properties of the curve:

1. The initial utility of income (i.e., the utility of the very first units of income) is infinite; and

2. No amount of income, however large, is sufficient to annihilate marginal utility of income.

These two statements are contested by Ricci. It will be very difficult, and somewhat cruel, to decide *by experiment* whether they are wrong or right! Of course, these questions only concern the results obtained by the statistical analysis. Frisch's *method* of measurement is quite independent of the above special form of the money utility curve.

We may now turn to the third step in static price and equilibrium theory: *the equalization of supply and demand* as one of the equations that determine price. In point of principle no objection can be made to this equation; at least in static theory. (In what respect it needs completion for dynamic problems will be considered further on.) It is a question, however, whether the solution of the equation for p will always be possible without difficulties. There may indeed be difficulties; on one side the number of possible solutions may be great, on the other side there may be no solution. The first difficulty, that of plurality of roots, has been mentioned by many earlier economists; in the last few years it has been treated in some detail by Lorenz and Koopmans.

Lorenz,[18] more in general, points to the large number of possible

[16] Cf. note 8.

[17] *Zeitschrift für Nationalökonomie*, II (1931), 625.

[18] "Denkrichtungen in der mathematischen Oekonomie und Statistik," *Jahrbücher für Nationalökonomie und Statistik*, CXXXVI (1932), 817.

roots and to the wide range of uncertainty arising from the uncertainty in many of the data. Koopmans[19] chooses some very much simplified examples in which only few subjects and few commodities play a rôle, the utility functions of these subjects in respect of these goods being quadratic—one of the simplest possibilities.

Another example of the difficulties that can present themselves in this field is signalized by Van Genechten[20] who states that Böhm-Bawerk's theory does not yield any solution for the rate of interest when the productivity function (the function relating productivity to length of roundabout way) is increasing beyond a certain tempo. This holds good, as Van Genechten points out, as long as "simple interest" is calculated—as is done by von Böhm—but calculating with "compound interest" eliminates the difficulty.

A third example of the same type is provided by Keynes[21] in his papers on the German transfer problem, when he points to the insolubility of the transfer problem—i.e., of the price problem for the German exchange rate—in one of two cases: (1) exports consist only of commodities of which output cannot be increased; or (2) exports consist only of articles having an elasticity of demand less than unity.

There are certain fields in general economics that are at present not so much in need of a broadening of the theoretical basis as in need of a minute working-out and concretization. Such problems present themselves in quite different divisions of economics. An example is given by a problem taken up by Engländer and studied in exact mathematical form by Frisch in a paper on a price theory with land and labor as factors of production.[22] The main difficulties in this problem consist in the characterization of the *different possibilities* and their mutual limits. For instance: when price A lies between certain limits 1 and 2, and price B between limits 3 and 4, product I will be chosen for production on a certain piece of land; when these prices lie between certain other limits, product II will be chosen, etc. These problems do not permit giving easily a summary of results; the survey can only mention them.

The same thing can be said of the newest development of those divisions of the theory of money, dealing with the velocity of circulation. Valuable work has been done by the more concrete estimations of that

[19] "De mogelijkheid van meervoudig economisch evenwicht," *De Economist*, 1932, pp. 679, 766, and 841.

[20] "Kritische Anmerkungen zur Zinstheorie," *Zeitschrift für Nationalökonomie* III (1932), 424, and *Verminderende Meeropbrengsten* (Haarlem, 1932), p. 163.

[21] "The German Transfer Problem," *Economic Journal*, XXXIX (1929), 1.

[22] "Einige Punkte einer Preistheorie mit Boden und Arbeit als Produktionsfaktoren," *Zeitschrift für Nationalökonomie*, III (1931), 62.

velocity by Keynes,[23] Holtrop,[24] Neisser,[25] and Marschak.[26] Distinction is made by Neisser and, with certain variations, also by the other writers, between several "spheres": (a) the "sphere of income," (b) the "sphere of enterprises" or "business sphere"; (c) the financial circulation; (d) what is called by Neisser "organwirtschaftliche Zirkulation." The general velocity of circulation is a weighted average of the partial velocities in each of these spheres, and is to be corrected for irregular hoarding and extraordinary reserves. Very roughly speaking, the partial velocity of circulation in the income sphere (or, better, in the different sub-spheres of income groups) is determined by the "income period"—the time elapsing between two subsequent income payments. As to the business sphere two viewpoints exist, each of which may apply to certain sub-spheres of this sphere. The so-called amonetary viewpoint pays most attention to the effects of discontinuous sales or discontinuous expenses. In these circumstances the "quant character" (to use a physical term) determines the amount of money necessary for a given transaction per year and hence the velocity of circulation. On the other hand, the so-called monetary conception is more valid for continuous production; it considers the "degree of differentiation" and the "capital intensity" of production as the most important factors. By the "degree of differentiation" is meant the average number of changes of hands commodities undergo before they reach the last market, the market on which consumers buy. In connection with this notion a distinction is made between (1) "velocity of circulation," indicating the proportion between all payments made per annum for commodities and services in the income and the business sphere and the stock of money, and (2) "efficiency" of money, indicating the proportion between annual income and stock of money. It is difficult, however, to describe adequately the results of newer investigations in these fields because the most important contributions are those that give the most detailed analysis.

II. DYNAMIC THEORY

Let us turn now to dynamic problems. According to the definition

[23] *A Treatise on Money*. London, 1930.

[24] *De omloopsnelheid van het geld*, Amsterdam, 1928; German: *Die Umlaufgeschwindigheit des Geldes, Beiträge zur Geldtheorie*, herausgegeben F. A. von Hayek, Vienna, 1933.

[25] "Der Kreislauf des Geldes," *Weltwirtschaftliches Archiv*, XXXIII (1931), 365. "Volksvermögen und Kassenbedarf," *Archiv für Sozialwissenschaft und Sozialpolitik*, LXIX (1933), 484.

[26] "Volksvermögen und Kassenbedarf," *Archiv für Sozialwissenschaft und Sozialpolitik*, LXVIII (1933), 385. "Vom Grössensystem der Geldwirtschaft," *Archiv für Sozialwissenschaft und Sozialpolitik*, LXIX (1933), 492.

of Frisch,[27] the distinction between statics and dynamics is not a distinction between two sorts of phenomena, but a distinction between two sorts of theories, i.e., between two ways of thinking. The phenomena may be *stationary* or *changing*, the theory (the analysis) may be *static* or *dynamic*. According to Frisch, an analysis may be called dynamic if one or more of the relations on which it is built contain variables that belong to different instants of time. In other words, a given equation must contain *at the same time* magnitudes belonging to different instants. A special case of this is obviously an equation that contains at the same time a certain variable and the rate of change with respect to time of this variable. Frisch concludes: "Every law in which derivatives of variables with respect to time and those variables themselves appear, is a dynamic law—*all other theoretical laws are static ones.*" I myself should prefer to include in dynamic laws also laws in which the connection with other instants is established in another way, for instance, by a simple time lag.[28] In the following remarks I suppose such connections between different time points to be also dynamic.

As is pointed out also by Frisch and more tacitly by several others, the kind of problem is changed by introducing dynamic relations. In static problems the solution consists in the indication of certain *values* for each of the economic phenomena considered. In dynamic problems, however, it is not one value that must be determined, but a whole range of values, relating to every elementary time period considered. In mathematical language, all economic phenomena shall be determined as functions of time.

The simple fact that a series of successive situations is considered in dynamic analyses introduces into such analyses a number of new elements. Since the fundamental notion in each economic analysis is the ophelimity or utility function, we can formulate our present subject in this way: what new variables can appear in this function as a consequence of a dynamic treatment of the problem?

In statics, the variables appearing in the utility function are income quantities of all sorts of goods and services. Those income quantities

[27] "Statikk og dynamikk i den okonomiske teori," *Nationaløkonomisk Tidsskrift*, LXVII (1929), 321.

[28] I quite agree to Tinbergen's formulation. The case of an equation connecting a variable and its rate of change with respect to time was intended as an example in the paper in *Nationaløkonomisk Tidsskrift*. I think this followed from the context, but I realize that the phrase quoted by Tinbergen may give rise to a misunderstanding. Later, in lectures at Yale and in Oslo (mimeographed), I have emphasized the fact that an equation connecting in any way whatsoever certain magnitudes that belong to different instants, may be looked upon as dynamic (or as cinematic, the difference between dynamics and cinematics is of no avail in the present connection).—EDITOR.

are not related to a certain time period, but are timeless. In dynamics the number of these variables is enlarged because, in principle, there may appear income quantities for each elementary time period in the future.

A first question of some interest is the question how many elementary time periods are represented in the utility function. In other words: *for how long a period of time does the subject in question look forward?* In newer literature this question has been mentioned several times without, however, leading to a systematic treatment.[29] The notion itself has been used by many older economists, e.g., in the discussions on Böhm-Bawerk's theories, etc., but it seems that a more concrete use of it has only been made in later times, for example, in the investigations by Roos and Evans.[30]

Besides income quantities, in dynamic problems "*stock quantities*" may figure in the utility functions. It is especially by the systematic dynamic treatment of problems that the difference between "stock quantities" and "income quantities" is now recognized in a much clearer fashion than in the more static way of thought; the use of the word "stock" often has been ambiguous. Those authors using mathematics have the opportunity to make a very sharp distinction, as this distinction has a form very appropriate for mathematical treatment; a stock of commodity is to be considered as an integral over time of the difference between production and consumption per time unit. The newer statistical investigations into the price formation of important commodities have shown, in addition, that stocks often are simply related to prices; and a further concretization of the theory of price formation has been possible.

Among the variables appearing in dynamic utility functions there are many relating to the future. Of these some may, indeed, be known with certainty. Most of them, however, will not be known, and it is only the *expectance* of a certain amount that can be considered by the subject or that, unconsciously, influences his acts. Such expectances—price as well as quantity expectances—play an important rôle in real phenomena, as was long since recognized. In the last few years, however, remarkable analyses have taken place in this respect.

Two problems, or complexes of problems, are especially interesting

[29] An attempt, not yet published, was made by Rosenstein-Rodan and discussed at the Lausanne meeting of the Econometric Society.

[30] Cf. e.g., C. F. Roos, "A Dynamical Theory of Economics," *Journal of Political Economy*, XXXV (1927), 637; "Dynamical Economics," *Proceedings of the National Academy of Science*, XIII (1927), 145; "A Dynamical Theory of Economic Equilibrium," *ibidem*, p. 280; G. C. Evans, *Mathematical Introduction to Economics*, 1930.

in this field. First of all, the relation that exists between expectances for the future and actual facts. Second, the relation that exists between expectances about a certain phenomenon and the same phenomenon as it develops in reality. The first question asks on what data "forecasts" are based, the second, whether the forecast has been reliable. The difference between forecast and reality is a measure of the risk connected with certain acts depending on that forecast.

Statistical investigations in this field are, in many respects, difficult, as a number of the factors influencing expectances are never published and may be of a personal character. In the field of commodity as well as stock speculation, however, groups of data are available and have been collected that are appropriate to further investigations. Especially has a lot of material been collected by the United States Grain Futures Administration. It seems possible to interpret some of the regularities found with the help of simple assumptions, which form an appropriate example of the sort of analysis meant here. These simple assumptions relate to the trade in non-perishable crops. Assuming that dealers include in their calculations a distinct number of years, e.g., 2 or 3, which we may call their horizon, and assuming further that their expectancy of future crops is, in accordance with the chance character of crops, that those shall be normal trend crops, it is possible[31] to account for the facts (1) that there is a good correlation between loco and future prices except in years of marked shortage, (2) that the annual fluctuations in sales of such products are proportional but not equal to the corresponding fluctuations in "total available stock" at the beginning of each crop year, and (3) that there is an inverse correlation between price and "stock at end of crop year."

As a suggestive minor result, one finds a relation between horizon and the coefficient of proportionality between sales fluctuations and crop fluctuations. Although the assumptions from which one starts can undoubtedly only be looked upon as first approximations, the cases considered seem to be clear examples of phenomena not to be understood by static methods only. A wide field seems open, also, in slightly different directions, for successful investigations as to a more systematic knowledge of the fundamentals for expectances.

As to the other problem of expectances, it is generally recognized that there regularly appear differences between reality and what it had been expected reality would be. This divergence is a measure of the amount of *risk* connected with enterprise. The influence of risk on the formation of prices and income has been dealt with more systemati-

[31] Cf. my paper: "Ein Problem der Dynamik," *Zeitschrift für Nationalökonomie*, III (1932), 169.

cally by e.g., Knight,[32] Hardy,[33] and Hicks.[34] It is, among other things, pointed out that although in a society in which risk exists a quasi-static equilibrium position may occur, there are systematic differences between such a state of things and an entirely static society, in which no risk occurs. New activities and new forms of income originate, and an absolute equilibrium no longer exists.

Another problem of great interest related to expectances, is the problem treated by Morgenstern[35]: assuming business forecast is possible, what will be the influence on business cycles? In contradiction to many older writers on business cycle prevention, Morgenstern fears forecasts will tend to intensify rather than to smooth out cycles. For important cases, for instance when a further fall in prices is forecasted, this must be admitted without any doubt; there seems, however, to be need for further investigation of this problem, as cases may also be indicated in which forecasts have other results.

There are also other elements that play a rôle in the variation of dynamic utility functions. Time may be *explicitly* included in the utility function, for the same reasons demand and supply functions may depend explicitly on time. Many suggestions have been given concerning this theme by the statistical derivations of seasonal changes, many of which are good examples of this sort of time dependency.

The functional relation between a daily "income" of a certain commodity and the degree of utility experienced by the subject having this income, may be analysed more in detail when account is taken of the fact that the satisfaction of this subject varies during each consumption cycle. Just before the consumption act the intensity of the want, "the craving" will be high, it will drop during the consumption act, and between these acts it will increase gradually. These problems are considered in detail by Frisch in his paper on dynamics.[36] In this study he arrives at the definition of marginal utility as a *functional* derivative, marginal utility itself being considered as a functional of the historical consumption curve. The most important part of this study is perhaps the distinction between what Frisch calls the "quenching curve" (the curve showing how the intensity of the want decreases during a rapid act of consumption), the "abstinence curve" (the curve showing the increase in intensity during a period where no consumption takes place), and the "maintenance curve" (the curve showing the connection between a given stationary level of consumption per unit

[32] *Risk, Uncertaity, and Profit*, 1921.
[33] *Risk and Risk Bearing.*
[34] "The Theory of Uncertainty and Profit," *Economica* (1931), p. 170
[35] *Wirtschaftsprognose*, Vienna, 1928.
[36] *Loc. cit.*, p. 349 ff.

time and the corresponding stationary level of intensity). He shows that two of these three curves are empirically *independent* data, the third depending on the two others in a mathematically well-defined way. For instance, from the fact that marginal utility is decreasing in the quenching sense it does *not* follow that it is decreasing also in the maintenance sense. There may perhaps be difference of opinion as to whether these problems are economic or physiological ones.

In dynamics, as in statics, a market theory may be built up starting from the notions of utility, and either the Walrasian or the polypolistic sorts of markets may be considered. We will not deal with these questions in detail; we shall follow up certain aspects of the question only. An interesting field in this connection is the study of certain forms of *technical conditions* that are of a dynamic nature. These equations express the connections laid between variables at different instants by purely technical (or also institutional) circumstances. The simplest example is that of a *constant time lag between the beginning and the end of a production process*. This fact has been placed in the centre of interest by statistical investigations in relation to business cycles and with supply analysis of agricultural products.[37] In many respects it seems to represent the simplest example of interrelations between two time periods. In a theoretical sense the problems related with such simple lags are interesting because they lead to oscillations in prices and production, some of which have been investigated statistically in a thoroughgoing way. A very interesting feature in the movements originating from the existence of lags is illustrated by Ricci.[38] Supposing for a moment that both demand and supply are linear functions of price, but that there exists a certain lag between price and the corresponding supply, the price movement of the commodity in question, resulting from any initial disturbance of equilibrium, may have different shapes, illustrated by Figure 2. Starting with a price OP_1, higher than equilibrium price, this price causes, after the given time lag, an offer of P_1P_2, to be found with the help of the supply curve S. This offer can only be sold at a price OP_4, to be found with the help of the demand curve D. After the time lag this gives a supply P_4P_5, etc. According to the quotient of the slopes of D and S, we get (1) when D is steeper than S, damped oscillations; (2) when D and S have the same slope, strictly periodical oscillations; and (3) when D is flatter than S, still wider oscillations. When demand and supply curves are not straight lines but general curves, more complicated movements are possible.

[37] Bean, "The Farmers' Response to Price," *Journal of Agricultural Economics* (1929), p. 368; Hanau, *Die Prognose der Schweinepreise*, and many others.

[38] "Die synthetische Ökonomie von Henry Ludwell Moore," *Zeitschrift für Nationalökonomie*, I (1930), 649.

The main theoretical interest seems to lie, however, in the fact that rather insignificant changes in the slope of demand or supply curves can disturb the stability of a market. There are other cases of a more complicated nature, but in some respects more in accordance with the economic system as a whole, that show the same property. One example of a somewhat more complicated nature was dealt with by myself; it related to a market of durable goods, where things are somewhat different. Here a slight variation can alter an oscillating movement into a one-sided movement. Other examples are given by Goudriaan[39] in connection with negatively sloped supply curves.

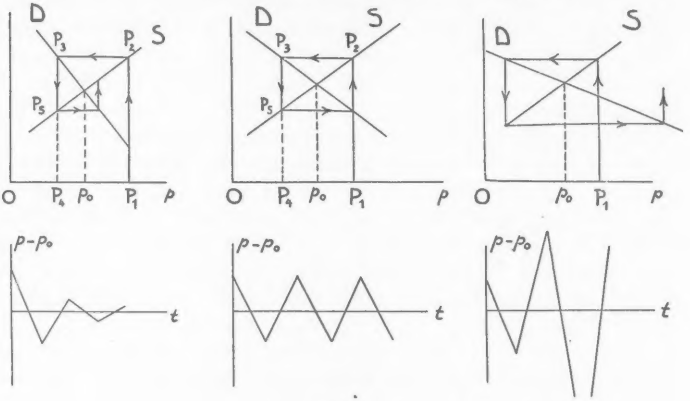

FIG. 2

Other examples of technical conditions binding variables of different time periods consist in the *long lifetime of many means of production* and some other commodities (furniture, clothing). This phenomenon introduces into the supply functions of such commodities integrals over longer periods giving an interesting mathematical feature to problems related to these markets. The same mathematical peculiarities appear when account is to be taken of the *buying power of those involved in a production process* lasting some finite time. This seems to have importance, especially in view of the newest development of business cycle theory and business cycle policy; but as this survey has to restrict itself to general economic theory, nothing will be said of those questions here.

A special case of a time lag is introduced when time is divided up into so-called "*income periods*," which are common now in the theory

[39] Cf. e.g., *How to Stop Deflation*, London, 1932.

of money. Although these periods are, on the whole, short in comparison to the lags dealt with above, the influence of their existence is important. This is shown in a convincing manner by a recent paper by Koopmans on the problem of "neutral" money.[40] The central problem dealt with by Koopmans may be re-exposed here. It relates to the question whether "neutral money" is identical with "stable money." A neutral monetary system is characterized by the property that all prices and exchanged quantities of commodities and services behave as if the Walrasian conditions for equilibrium were fulfilled. In particular, income and outlay shall be equal at each moment, so that, *in summa*, no hoarding or creating of money is permitted. According to Koopmans, in a neutral monetary system, changes in the general price level are possible, whereas this is contested by other writers. To prove his statement he considers a society in which neutral monetary conditions are fulfilled and in which by a change in efficiency (or yield per acre for agricultural products) the price of one commodity A changes. Owing to the "doctrine of compensating price changes" other prices will move in the opposite direction and it will be possible to choose an adequate index of prices that will show no change. This doctrine is made plausible by its defenders by pointing to the buying power consumers of A will save (in the case of a fall of A-prices) or will be short of (in the case of a rise in A-prices).

On the other side there is a "doctrine of cumulative price changes" arguing that other prices will move in the same direction as A-prices; its adherents pay special attention to the buying power of those producing A.

A first interesting point in this controversy is, as Koopmans remarks, that results depend entirely on the assumption that the elasticity of demand for A is lower than 1. Suppose this elasticity to be higher than 1, and the results are quite the reverse! And the case of an elasticity equal to 1 is very appropriate to show that compensation is not necessary, for in that case no other price changes, and therefore the general index number of prices has to change in the same direction as A-prices. In principle this furnishes the proof that there is no form of index number for which in all cases neutral money is equivalent to stable money.

It is, however, interesting to follow Koopmans in his further expositions, as these will show us the influence of the time lag referred to.

Supposing that elasticity of demand for A be not equal to 1, he states that the increase in buying power of A-producers is always com-

[40] *Zum Problem des "neutralen" Geldes, Beiträge zur Geldtheorie,* herausgegeben von F. A. Hayek, Vienna, 1933.

pensated for by the decrease of buying power of A-consumers. (Increase as well as decrease may also be negative.) And so for those cases it is also easily proved that no other price changes occur. It is now interesting to confront this with the opinions of Stucken, Mahr, and Hawtrey, who deny the compensation just mentioned. Instead of that, they accept a sort of "law of compensating price sums," starting from the idea that money can be spent only once. The income earned in the last income period of the original status will be spent in the next period. When, then, A-prices have changed, and also A-quantities, the total sum spent on A will have changed, as soon as elasticity is either lower or higher than one. Suppose it has risen, then the total buying power available for other commodities will have decreased and their price level will fall. This means compensating movement when the change in A-price was a fall, and cumulative movement when this change was a rise.

This seeming contradiction arises from the fact that it has been supposed that in the first income period of the "new status" A only spends an income equal to that earned in the earlier periods. Had he been already spending in this first period an income corresponding to the new state of affairs, things would have taken their course as pointed out by Koopmans; and only the latter is in accordance with "neutral" money conditions. The picture given by the other writers may be far more realistic, Koopmans adds, but it is not a "neutral" situation. The lag appearing between A's earning of a certain income and spending it, is characteristic for a non-neutral monetary system.

The importance of this exposition is particularly clear when we continue the argument for some further income periods. The second income period after the original state of things shows, indeed, a larger buying power of A coming on the market; but, in addition, the other producers have now a smaller buying power just compensating A's. Suppose the original buying power spent per unit of time by all producers to have been K; originally the price of A may have been p_1, the quantity produced and consumed a_1, after the change these quantities may be p_2 and a_2.

In the first income period after the change there is spent: by A-producers, a_1p_1; by others, $K-a_1p_1$; in total, K. From this buying power, however, now a_2p_2 (supposed $>a_1p_1$) is spent on A, therefore $K-a_2p_2$ on other commodities.

In the second income period there will be spent by A-producers a_2p_2, and by others $K-a_2p_2$, total, K. From this, again a_2p_2 will be spent on A, the rest on the other products. So the situation *remains different* from the situation created with a neutral monetary system.

In summary, under the régime of neutral money only the price of A changes, other prices remain constant, the velocity of circulation of money has increased; under the prevailing monetary régime other prices may have compensated for A's price movement, and, in that case, the velocity of circulation has remained constant—but, depending on the elasticity of demand for A, also cumulative price movements are possible with a corresponding change in velocity of circulation.

An interesting use, from the theoretical viewpoint, of the nature of lags is made by Hayek[41] in his newest theory of business cycles. I will not discuss in general his theory here, as this belongs to the subject of another survey, but his leading idea has importance for other divisions of theory also. Hayek points to the effects of an increasing "roundabout way of production" in the sense of Böhm, caused, as Hayek assumes, by the rising difference between the "real" and the "money" rate of interest. These effects consist in a certain shortage of products during the transition period, in which—to state the case with some exaggeration—products from the earlier production process may be said already to be consumed, and that of the later not yet deliverable. Here we have a problem related essentially to *varying* lags.

Varying lags are also introduced when the influence of a phenomenon A on another phenomenon B is of a cumulative nature, as is suggested by Hamburger and Le Corbeiller,[42] who advocate the application of the theory of relaxation oscillations in economics. A simple case may be stated here for illustration. The influence of employment on wages may be considered as a relation with a constant lag between those two phenomena (trend movements eliminated). It may, however, also be looked upon as the consequences of a direct relation between employment and the velocity of movement of wages; and for a psychological interpretation it may be more natural to start from this latter relation. The effect of such a relation is, however, that wages are now lagging in a more irregular way after employment, as soon as employment itself shows deviations from a sine curve or from an exponential curve.

In newer literature some problems of a semi-theoretical character have been treated which, both as concerns the method used and their great practical importance, deserve attention. I mean the sort of prob-

[41] Hayek, *Preise und Produktion*, Veröffentlichungen des Oesterreichischen Instituts für Konjunkturforschung, Vienna, 1931.

[42] "Een nieuwe weg voor conjunctuuronderzoek, een nieuwe richtlijn voor conjunctuurpolitiek," *De Economist*, LXXIX (1930), 1; also: "Analogie des fluctuations économiques et des oscillations de relaxation," *Indices du Mouvement des Affaires*, IX (1931), Supplement.

lem dealt with, e.g., by Kahn.[43] The question is: given a situation of unemployment, the unemployed receiving a "dole," what will be the effect of supplementary investments (e.g. in the form of "public works" by governments) on unemployment? Kahn distinguishes three sorts of laborers coming into work again: (1) those employed in the public works, (2) those producing materials, machinery, etc., to be used in the public works, and (3) those employed because the first two groups expand their consumption by an amount, the value of which corresponds to the difference between wages and dole; those getting employed in the same way because of the group just mentioned having got work, etc. This last group is estimated by Kahn in a first approximation by the following argument. Let the difference between wages and dole be k times the cost associated with the supplementary employment of one laborer; k will be less than unity. Then, those mentioned first in the third group—getting their job by groups 1 and 2—will be k times groups 1 and 2; those mentioned in the second place in group 3 will be k^2 times groups 1 and 2, and so on. In total group 3 will contain

$$k+k^2+k^3 \cdots = \frac{k}{1-k}$$

times groups 1 and 2. Kahn estimates, with the help of data that we need not repeat here, that $\frac{k}{1-k}$ may be between 0.5 and 1; which is a very important result for practical purposes. It is not difficult to criticize this very simple method—also the author is aware of the approximate character of the whole method—it is far from easy, however, to indicate the limits of this approximation. As the author observes, the length of the income period (in the sense used formerly) is neglected; this is one of the reasons why the formula would give far too great values in the case that, for instance, the dole was zero.

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[43] "The Relation of Home Investment to Unemployment," *Economic Journal*, XLI (1931), 173.

MAXIMUM PRODUCTION STUDIED IN A SIMPLIFIED ECONOMIC SYSTEM

By GRIFFITH C. EVANS

Vice-presidential address, given under the title "The Theory of Money," for Section K of the American Association for the Advancement of Science, at a joint meeting with the Econometric Society and the Mathematical Association of America, Dec. 27, 1932.

1. *Introduction.*—Large numbers of simultaneous equations in a large number of variables convey little information, whether they happen to be correct or not, about an economic system. They may serve as an expression of a theory which cannot be adequately described in words, but, unless they offer some opportunity for numerical control, provide no check on the theory itself. There are, however, two possibilities open in the direction of increasing information, first, to obtain, by statistical methods, statements about averages or sums which have a significant relation to the system, or, otherwise, to describe a second system which consists of only a few variables, subject to relations which are suggested by experience. The second alternative is doubtless the simpler. In some sciences both methods find application. In the physical theory of gases, the experimentally born concepts of pressure and temperature obey simple laws, in approximation, and turn out to be sums and averages in kinetic theory, where a fuller and more systematic treatment may be found for them.

Similar comprehensive variables may be introduced into economic theory. Capital, labor, production, and money are, in fact, talked of as such, and general statements are made about them, for example, by socialists and conservatives, without analyzing them into all their multitudinous parts. It is a tempting project therefore to see if one can build a theory in terms of these few variables. Of course, during the process, some analysis of the underlying concepts will be necessary.

2. *General variables.*—We write

(2.1) U_1 = production per unit time of capital, or services of capital,
U_2 = services per unit time of labor,
U_3 = production per unit time of commodity for direct consumption.

We assume that there are no stocks or stores of material being built up or drawn on, but that there is a steady state with as much used as produced. Hence, the capital services will be divided into two portions, used respectively in the production of capital services themselves and in the production of commodity. Likewise labor is wholly used for

the production of capital and commodity, and the commodity is divided into three parts, for direct consumption by the owners of the three services. In symbols,

$$
\begin{aligned}
U_1 &= U_1^{(1)} + U_1^{(3)} \\
U_2 &= U_2^{(1)} + U_2^{(3)} \\
U_3 &= U_3^{(1)} + U_3^{(2)} + U_3^{(3)}.
\end{aligned} \tag{2.2}
$$

If further, we write

$$U_3^{(1)} = aU_3, \quad U_3^{(2)} = bU_3, \quad U_3^{(3)} = cU_3,$$

it follows from (2.2) that

$$a + b + c = 1. \tag{2.3}$$

Capital and labor are combined in various ways, as factors of production, in the production of commodity and capital. The combination is not necessarily invariable, because the technique of manufacture allows for the relative advantages of using more capital, or more labor. With a given amount of labor, there would be used a definite amount of capital services to produce a given amount of product. We may express this freedom of choice in terms of certain formulae of production

$$U_1 = \phi(U_1^{(1)}, U_2^{(1)}), \quad U_3 = \theta(U_1^{(3)}, U_2^{(3)}). \tag{2.4}$$

We shall need to return to these functions later for a slightly more careful description.

For simplicity, we make a special hypothesis about labor, namely that

$$U_2 = l, \textit{ a constant.} \tag{2.5}$$

This hypothesis is not unreasonable, if one abandons the "living wage" theory, which assumes that there is a large and variable portion of labor which remains unemployed. Except during times like the present, when there is a large fraction of every class which remains unproductive, it seems possible to adopt this steady employment hypothesis. It is expounded in detail by Divisia.[1]

The relations so far described limit the freedom of the variables, but do not determine them. In order to obtain definite values for them, we need to make some hypothesis which will give its typical character to the system. Here we shall assume what seems to have been characteristic of our industrial apotheosis, namely, that the production for direct consumption shall be a maximum.

[1] F. Divisia, *Économique rationnelle* (Paris, 1928), Ch. x.

(2.6) *U_3 to be maximum.*

By equations (2.2), (2.4) and (2.5) we have

$$U_1^{(1)} + U_1^{(3)} = \phi(U_1^{(1)}, U_2^{(1)})$$

$$U_3 = \theta(U_1^{(3)}, l - U_2^{(1)})$$

where $U_1^{(1)}$, $U_1^{(3)}$, $U_2^{(1)}$ may be regarded as independent variables. In order to simplify the notation, let us denote them by x, y, z respectively; so that we have

$$x + y = \phi(x, z) \tag{2.7}$$

$$U_3 = \theta(y, l - z) \tag{2.8}$$

where U_3 is to be a maximum, subject to the relation (2.7).

At a first glance, one is tempted to say that y, z may be regarded as independent variables, and x given by (2.7); and the maximum may be obtained simply by writing

$$\theta_1(y, l - z) = 0, \qquad \theta_2(y, l - z) = 0,$$

where θ_1 and θ_2 are the partial derivatives of θ with respect to its two arguments in (2.4). That is,

$$\theta_1(\alpha, \beta) = \frac{\partial\theta(\alpha, \beta)}{\partial\alpha}, \qquad \theta_2(\alpha, \beta) = \frac{\partial\theta(\alpha, \beta)}{\partial\beta}.$$

Thus we obtain two equations to determine y, z, and (2.7) gives us x. But this sophomoric result is illusory.

In the first place, we are interested merely in positive values of all the quantities encountered, so that the range of the variables is restricted. In the second place, the functions ϕ and θ turn out, on slight examination, not to fit with the above kind of determination; for they are evidently to be taken as increasing functions of their arguments, and therefore these arguments will not in general have values which will make any partial derivative zero. We must instead adopt the alternative of making U_3 a maximum by finding, in some way, the largest available values of y, $l-z$.

If we take $z=0$, which gives the largest value of the argument $l-z$, equation (2.7) becomes

$$x + y = \phi(x, 0).$$

But in this case we have the production of capital without labor (see (2.4)), and the resulting amount of capital produced must be approximately merely that which we have already. Hence the above relation reduces to $x+y=x$, whence $y=0$. But now we have the production of commodity without capital, in the equation

$$U_3 = \theta(0, l)$$

and this, in the practical case, is not a maximum value. Similarly, to put $z=l$ makes $x+y$ large, but puts us in the position of producing commodity without labor, and, with our as yet only semi-perfect technique, makes U_3 small. Accordingly we seek the proper intermediate value of z.

The hypotheses on the functions ϕ, θ are that

$$\phi(x, 0) = x, \quad \phi(0, z) = 0, \quad \theta(0, l - z) = 0, \quad \theta(y, 0) = 0,$$

that both functions are increasing functions of their arguments, and finally that, for any z, $\partial\phi/\partial x$ is >1 until a critical value of increasing x, whereupon $\partial\phi/\partial x = 1$, this value depending on z. In this last hypothesis, we are assuming that, for a given amount of labor used in producing capital, the amount of capital produced increases more rapidly than the amount of capital used in producing it, until a critical value, when marginal doses of capital merely reproduce themselves. After this value there may even be a relative loss.

We write

$$y = \phi(x, z) - x$$

and for a given z we determine x so that y will be a maximum. This condition is expressed by the equation $\partial y/\partial x = 0$, or

$$\phi_1(x, z) = 1, \tag{2.9}$$

which we use to find x in terms of z, and thus also y in terms of z. In this way θ becomes a function of z alone. Since, as we have seen, it has a maximum for some value of z between 0 and 1, this value of z will satisfy the equation

$$\frac{d\theta}{dz} = 0.$$

But

$$\frac{d\theta}{dz} = \theta_1 \frac{dy}{dz} + \theta_2 \frac{d(l - z)}{dz}$$

and from (2.7), (2.9)

$$\frac{dy}{dz} = \phi_1 \frac{dx}{dz} + \phi_2 - \frac{dx}{dz} = \phi_2$$

so that $d\theta/dz$ reduces to $\theta_1\phi_2 - \theta_2$. Hence at the optimum point we have the equation

$$\theta_1(y, l - z)\phi_2(x, z) - \theta_2(y, l - z) = 0. \tag{2.10}$$

The equations (2.7), (2.9), (2.10) provide three simultaneous equa-

tions to determine the three unknowns x, y, z. The quantity U_3 is then given by (2.8).

If we express our problem in terms of differentials, the solution which has just been obtained appears in an interesting manner. The equations (2.7), (2.8) yield the relations

$$(\phi_1 - 1)dx - dy + \phi_2 dz = 0$$
$$d\theta = \theta_1 dy - \theta_2 dz = 0,$$

and we may regard the problem as solved formally by the use of a Lagrangian multiplier λ. We obtain

$$0 = \lambda(\phi_1 - 1)dx + (\theta_1 - \lambda)dy + (-\theta_2 + \lambda\phi_2)dz = 0,$$

or

$$\lambda(\phi_1 - 1) = 0, \quad \theta_1 - \lambda = 0, \quad -\theta_2 + \lambda\phi_2 = 0.$$

The sophomoric solution is obtained by putting $\lambda = 0$. The other possible choice of λ, namely, $\lambda = \theta_1 \neq 0$, yields the equations (2.9), (2.10).[2]

3. *Coefficients of distribution.*—Our system may now be modified in various ways in order to make it correspond to different interesting types of social order. Before proceeding, *gradus ad obscuritatem*, however, it may be opportune to make one or two observations with regard to the coefficients a, b, c. These represent the fractions into which the product U_3 is divided for the respective three classes of the society. They have not been determined by the theory, and so remain entirely

[2] By calculating

$$\frac{\partial^2 U_3}{\partial x^2}\frac{\partial^2 U_3}{\partial z^2} - \left(\frac{\partial^2 U_3}{\partial x \partial z}\right)^2,$$

and writing $\phi_1 = 1$, we find that sufficient conditions that the given position yield a maximum are that

$$\theta_1\phi_{11} = \frac{\partial^2 U_3}{\partial x^2} < 0$$

$$\theta_{11}\theta_1\phi_{11}\phi_2^2 - 2\theta_{12}\theta_1\phi_{11}\phi_2 + \theta_{22}\theta_1\phi_{11} + \theta_1^2(\phi_{22}\phi_{11} - \phi_{12}^2) > 0$$

conditions which are satisfied, when $\theta_1 > 0$, $\theta_2 > 0$, $\phi_1 > 0$, $\phi_2 > 0$, if

$$\phi_{11} < 0, \ \phi_{22}\phi_{11} - \phi_{12}^2 > 0$$
$$\theta_{11} \leqq 0, \ \theta_{22} \leqq 0, \ \theta_{12} \geqq 0.$$

The last set of conditions are eliminated if θ is linear.

We may write, as approximations in the neighborhood of the position of equilibrium,

$$x + y = \phi(x, y) = -ax^2 - bz^2 + cxz + kx + m$$
$$U_3 = \theta(y, z) = ry + s(l - z) + e, \ r > 0, \ s > 0,$$

and obtain $\phi_{11} = -2a$, $\phi_{22} = -2b$, $\phi_{12} = c$, so that these conditions are satisfied if $a > 0$, $4ab > c^2$. The equilibrium position is easily calculated, and k, m, e may be given so that the solutions are positive.

arbitrary, subject to whatever hypotheses it may be desirable to adjoin. In a simple system, which we may call *communistic*, for lack of a better term, they may be assigned arbitrary constant values. In a system in which financial structure is paramount, they may be determined by complicated relations of prices. We shall see something of this later on. For the present we continue without the complication of money.

4. *Quantities variable with time.*[3]—Without difficulty, the element of time may be introduced into the equations, producing modifications which it is desirable to consider, since the process of manufacture and bringing to market is not instantaneous. For simplicity, it may be assumed that there is a constant time interval T between the moment when the factors of production are engaged and the moment when the products are released. We indicate time by a superscript, as follows:

$$l = l(t), \quad l' = l(t - T), \quad l'' = l(t - 2T), \text{ etc.},$$

in order to avoid complicated parentheses.

Hypotheses may now be written down to describe the system. We suppose (2.2), (2.3) to remain valid at the variable time t, and rewrite (2.7), (2.8) in the form

$$x + y = \phi(x', z') \tag{4.1}$$

$$U_3 = \theta(y', l' - z'). \tag{4.2}$$

It is desirable for us to state our maximizing assumption in such a way as to get a continuing process, rather than one centering on the particular instant t. Hence we shall try to determine the variables so as to make the total production a maximum over a long period of time; that is

$$I = \int_a^b U_3(t)dt = \int_a^b \theta(y', l' - z')dt$$

is to be a maximum, by proper choice of the variable functions, subject to (4.1).

We make x, z rather than y, z the independent variables, which is essentially the process used before for (2.7), (2.8), and obtain the quantity I in terms of these variables. From (4.1)

$$I = \int_a^b U_3(t)dt = \int_a^b \theta(\phi(x'', z'') - x', l' - z')dt \tag{4.3}$$

When we replace x by $x+\epsilon\psi$, z by $z+\eta\rho$ (ψ and ρ being arbitrary functions of t, and ϵ and η arbitrary parameters independent of t), and thus x' by $x'+\epsilon\psi'$, x'' by $x''+\epsilon\psi''$, etc., we obtain the conditions

[3] The reader may omit Art. 4 without loss of understanding of Art. 5.

$$\left(\frac{\partial I}{\partial \epsilon}\right)_{\epsilon=0,\,\eta=0} = 0, \qquad \left(\frac{\partial I}{\partial \eta}\right)_{\eta=0,\,\epsilon=0} = 0$$

in the following form, by differentiation under the integral sign:

$$\begin{aligned} &\int_a^b \theta_1(\phi(x'', z'') - x', l' - z')\,[\phi_1(x'', z'')\psi'' - \psi']dt = 0 \\ &\int_a^b \{\theta_1(\phi(x'', z'') - x', l' - z')\phi_2(x'', z'')\rho'' \\ &\qquad - \theta_2(\phi(x'', z'') - x', l' - z')\rho'\}dt = 0. \end{aligned} \tag{4.4}$$

If I is to be a maximum for the chosen functions x, z, considering all variations of them, the above equations must be satisfied for arbitrary functions ψ, ρ. In order to see what this fact implies, we change the variable of integration in the first term from t to $t-T$ and rewrite the first equation in the form

$$\int_{a-T}^{b-T} \theta_1(\phi(x', z') - x, l - z)\phi_1(x', z')\psi' dt - \int_a^b \theta_1(\phi(x'', z'') - x', l' - z')\psi' dt = 0$$

or

$$\begin{aligned} &\int_a^{b-T} \{\theta_1(\phi(x', z') - x, l - z)\phi_1(x', z') - \theta_1(\phi(x'', z'') - x', l' - z')\}\psi' dt \\ &+ \int_{a-T}^{a} \theta_1(\phi(x', z') - x, l - z)\phi_1(x', z')\psi' dt \\ &- \int_{b-T}^{b} \theta_1(\phi(x'', z'') - x', l' - z')\psi' dt = 0. \end{aligned}$$

None of the intervals $(a-T, a)$, $(a, b-T)$, $(b-T, b)$ overlap, and in each of them $\psi' = \psi(t-T)$ is arbitrary. Hence the integrands must vanish. Of the three resulting equations we are interested merely in the first:

$$\theta_1(\phi(x', z') - x, l - z)\phi_1(x', z') - \theta_1(\phi(x'', z'') - x', l' - z') = 0, \tag{4.5}$$

for we are looking for a steady condition rather than one which refers to intervals at the ends of (a, b), and we have made no hypothesis about conditions outside of (a, b). Or else, from a slightly different point of view, we get (4.5) alone as the condition if we let a tend to $-\infty$, b to $+\infty$.

The second of equations (4.4) may be treated in the same manner, and we obtain the equation

$$(4.6)\quad \theta_1(\phi(x', z') - x, l - z)\phi_2(x', z') - \theta_2(\phi(x'', z'') - x', l' - z') = 0.$$

This equation is to be solved with (4.5) in order to determine x, z at the time t in terms of their values at previous times.

But this in general cannot be done directly, since x, z at the time t enter into both equations only through the same combination. In order to obtain a more serviceable second equation, we eliminate θ_1 from these, and in the resulting equation, which is of the form $\phi_1\theta_2 - \phi_2\theta_1 = 0$, advance the times by T, that is, replace $t-T$ by T, $t-2T$ by $t-T$, etc. Recalling (4.1), we have accordingly the three equations

$$\begin{gathered}\phi_1(x, z)\theta_2(\phi(x', z') - x, l - z) = \phi_2(z, z)\theta_1(\phi(x', z') - x, l - z)\\ (4.7)\quad \phi_1(x', z')\theta_1(\phi(x', z') - x, l - z) = \theta_1(\phi(x'', z'') - x', l' - z')\\ y = \phi(x', z') - x.\end{gathered}$$

We may regard these equations as solved for x, y, z, so that we obtain solutions of the form

$$\begin{aligned}x &= X(x', z'; x'', z''; l, l')\\ (4.8)\qquad z &= Z(x', z'; x'', z''; l, l')\\ y &= \phi(x', z') - X(x', z'; x'', z''; l, l').\end{aligned}$$

Hence if x, z are given arbitrarily as functions of the time, in an interval of time from t_0 to t_0+2T, they are, with y, determined at all later times.

In particular, if the amount l of available labor is constant, and if x, z are maintained as constant in each of the intervals (t_0, t_0+T) (t_0+T, t_0+2T), the x, y, z will have constant values in each of the successive intervals of time of length T. A further hypothesis of continuity will lead the problem back to that one which was earlier treated. But in the general case the point of view is one of dynamics.

5. *Prices and minimum cost.*—The results are not altogether what might be expected when money is introduced into the system, even although we remember that our main hypothesis deals with the amount of U_3 produced, rather than with profits. We retain the hypotheses (2.7), (2.8), relative to the factors of production,

$$x + y = \phi(x, z), \qquad U_3 = \theta(y, l - z)$$

with

$$l = U_2, \quad x = U_1^{(1)}, \quad y = U_1^{(3)}, \quad z = U_2^{(1)}$$

and introduce prices p_1, p_2, p_3 of U_1, U_2, U_3 respectively.

The assumption of a steady state with respect to money, that is, that money is not hoarded up in any compartment, yields the equations (selling value = cost[4] plus cost of U_3 consumed)

$$\begin{aligned} (x+y)p_1 &= xp_1 + zp_2 + ap_3U_3 \\ lp_2 &= bp_3U_3 \\ U_3p_3 &= yp_1 + (l-z)p_2 + cp_3U_3, \end{aligned} \tag{5.1}$$

where a, b, c are the distribution coefficients for the allocation of U_3, as yet undetermined, and as before

$$a + b + c = 1.$$

The equations (5.1) are not independent, since an identity is obtained by adding them together. We shall use therefore merely the first two of them, in the form

$$\begin{aligned} ap_3U_3 &= yp_1 - zp_2 \\ bp_3U_3 &= lp_2 \end{aligned} \tag{5.2}$$

It is natural to assume, in a competitive system, that for given prices p_1, p_2, p_3 the variables x, z shall be chosen so as to make the costs of manufacture of $U_1 = x+y$ and U_3 a minimum. These costs are

$$q_1 = p_1x + p_2z, \qquad q_3 = p_1y + p_2(l-z).$$

For given amounts of $U_1 = x+y$, U_3, from (2.7), (2.8) and the above equations for q_1 and q_3, we have

$$\begin{cases} dx + dy = 0, \qquad \phi_1dx + \phi_2dz = 0 \\ dq_1 = p_1dx + p_2dz = 0 \end{cases} \tag{5.3}$$

$$\begin{cases} \theta_1dy - \theta_2dz = 0 \\ dq_3 = p_1dy - p_2dz = 0 \end{cases} \tag{5.4}$$

so that

$$\frac{p_1}{p_2} = \frac{\phi_1(x,z)}{\phi_2(x,z)} = \frac{\theta_1(y,l-z)}{\theta_2(y,l-z)}. \tag{5.5}$$

Whether we consider (5.3), (5.4) as holding simultaneously for the same dx, dy, dz, or (5.3) holding for one set of values of differentials, and (5.4) for another set, does not make any difference mathematically, on account of the separation of the variables. Yet the hypotheses in

[4] In the first equation (5.1) the term xp_1 is used, rather than x multiplied by the average unit cost of manufacture of x, since each individual does not manufacture his own capital, but buys it. Similarly the term cp_3U_3 is used in the third equation (5.1).

the two cases are different, although both deal with minimum cost; for the first describes that kind of competition where the producers of one kind of service estimate the production of the others and restrict the nature of their competition, while the second describes an unlimited competition in which each blindly considers his own case without reference to the others. With more variables the difference is more sharply brought to notice.

Subject to (5.5) we wish to make U_3 a maximum. Writing

$$\mu = \frac{p_1}{p_2}, \qquad \lambda = \frac{a}{b},$$

we have from (5.2) the relation

$$\lambda = \frac{1}{l}(\mu y - z), \tag{5.6}$$

giving λ in terms of μ, y, z. But we may regard the equations

$$x + y = \phi(x, z), \qquad \mu = \frac{\phi_1(x, z)}{\phi_2(x, z)} = \frac{\theta_1(y, l - z)}{\theta_2(y, l - z)}$$

as solvable for x, y, z in terms of μ. Hence by substitution in $U_3 = \theta(y, l-z)$, that quantity is obtained as a function of μ alone, and we choose μ so as to make U_3 a maximum. Hence we may regard the problem as solved as far as the quantities $x, y, z, U_3, \lambda, \mu$ are concerned.

It is an important fact that the maximum of U_3 thus obtained is the same as that already obtained in the simpler problem of Art. 2, without prices. For if U_3 is to be a maximum, even with respect to certain constrained changes of the variables, we must still have for those changes the relations that derive from (2.7), (2.8), namely

$$dx(\phi_1 - 1) - dy + \phi_2 dz = 0$$
$$\theta_1 dy - \theta_2 dz = 0,$$

and if the first of these is multiplied by θ_1 and added to the second times ϕ_1 there results the equation:

$$\theta_1(\phi_1 - 1)(dx + dy) + (\theta_1\phi_2 - \theta_2\phi_1)dz = 0.$$

But $\theta_1\phi_2 - \theta_2\psi_1 = 0$, in virtue of (5.5). Whence since θ_1 is not zero, it follows that $\phi_1 = 1$, as in Art. 2. Now, however, we have the two equations (2.9), (2.10) of Art. 2.

In other words, the constraint introduced by the hypothesis of minimum costs does not affect the maximum of production, or the determination of the variables x, y, z. But it does affect the distribution of U_3 for consumption, since the ratio λ is given in terms of μ, which is de-

termined by (5.5). If however the state of the system were to be determined by the maximizing of some function of the variables, other than U_3, we should not, in general, obtain the same determination when prices were introduced.

A further hypothesis must be made to determine the ratios c/b, p_3/p_2. It is natural to make an assumption with regard to the relation between the compartments (1) and (3), whose members are to a large extent interchangeable. From this point of view p_3/p_1 may be regarded as known, and in fact we may assume that with suitable choice of units the two are equal, and write $p_3=p_1$.

Instead of making U_3 a maximum and assuming a known relation between p_1 and p_3, we might have chosen some other quantity which involves U_3 and the ratios of prices $\mu=p_1/p_2$, $\nu=p_3/p_2$, and thus determine U_3, μ, ν, a, b, c altogether. Or, to make a slight digression, let us consider an intermediate situation, in which costs are a minimum and cU_3 (that is, the portion of U_3 which is allotted to the producers of U_3) a maximum. In this case the equilibrium values of the variables are in part on the boundary of their region of definition, instead of entirely in the interior. In fact, we obtain evidently the formal result $a=0$, $b=0$, $c=1$, with U_3, μ, λ determined as before; and we fall back on the theory of the living wage. That is, assuming an irreducible minimum β for bU_3, in the equilibrium position we have

$$b=\frac{\beta}{U_3}, \qquad a=\frac{\lambda\beta}{U_3}, \qquad c=1-(1+\lambda)\frac{\beta}{U_3},$$

with U_3 determined as before.

6. *The equation of exchange.*—With the ratios of prices already determined, we are now in a position to determine the prices themselves, by means of the "equation of exchange"[5]

$$p_1U_1+p_2U_2+p_3U_3=MV, \tag{6.1}$$

where M is the amount of money in circulation, and V is the velocity of circulation, that is, the number of times one unit of currency is spent during the year. The equation (6.1) is a sort of Boyle's law for finance. It is a "truism," so-called, in the same sense that Boyle's Law is a truism in the kinetic theory of gases—likewise, however, a significant relation if M and V can be calculated.

The velocities of circulation for the various classes of individuals of the community depend upon the number of times the individuals of these classes receive income during the year, and the nature of the

[5] I. Fisher, *The Purchasing Power of Money* (New York, 1922), Ch. II, III; F. Divisia, *loc. cit.*, Ch. XIV, XIX.

income. If the income is spent at a uniform rate, while it is received in k_i equal payments, for the class (i), the average amount in (i) is $m_i=p_iU_i/(2k_i)$, and the corresponding velocity of circulation is $V_i=p_iU_i/m_i=2k_i$. Regarding then these velocities of circulation as known, the equation (6.1) yields the relation

$$M = m_1 + m_2 + m_3 = \frac{p_1U_1}{V_1} + \frac{p_2U_2}{V_2} + \frac{p_3U_3}{V_3}, \tag{6.2}$$

where M is known, $U_1=x+y$ and $U_2=l$. Thus (6.2) gives us the additional relation necessary to determine the prices.

The nature of M may be scrutinized more in detail. If M is merely money in the strict sense, its value is known by means of information from the sources of coinage and the amounts of savings deposits. In practice, however, M includes also the total of bank deposits subject to check. We write

$$M = M_0 + M',$$

distinguishing from each other these two media of exchange, and note that M' is an amount that is secured by lien on capital. What is important here is the ratio α of M', to p_1U_1, which is the gross revenue from capital, or the ratio γ of M' to $p_1U_1-p_1x-p_2z\equiv p_1y-p_2z$, which is the net revenue from capital. If we write $p_3=p_1$, $V_3=V_1$, and as a first approximation take $V_2=\infty$, recognizing the fact that monthly wages are spent rapidly and weekly wages received in small amounts (6.2) becomes in the first case, setting $M'=\alpha p_1U_1$,

$$p_1\left\{U_1\left(\frac{1}{V_1}-\alpha\right)+\frac{U_3}{V_1}\right\} = M_0, \tag{6.3}$$

and, in the second case, setting $M'=\gamma(p_1y-p_2z)$, $U_1=x+y$,

$$p_1\left\{\frac{x+y+U_3}{V_1}-\gamma y\right\} + p_2\gamma z = M_0. \tag{6.4}$$

A statistical analysis of these relations might be interesting.

7. *Production lag and a money system.*—The fact that the manufacturing process takes time has been made the basis of theories of crises and business cycles, and the dangerous possibilities of such events were pointed out in these theories before the present catastrophe. Instability and the credit system are not easy to separate. In this paper, however, such questions are not considered, since our hypothesis that all production is consumed virtually eliminates them.

In this section of the paper we restrict ourselves to showing that the introduction of money relations into the dynamical system of Art. 4

has no more effect than it had in the simpler system of Art. 2. It helps to determine the distribution of commodities of direct consumption but not the total amount of their production.

We retain the equations (4.1), (4.2), with the notation employed in Art. 4, and, as there, make maximum the total production of U_3 from $t=a$ to $t=b$:

$$I = \int_a^b U_3 dt = \int_a^b \theta(y', l' - z')dt;$$

but we do so now only considering those values of the variables which make the costs of manufacture of given amounts of $U_1=x+y$ and U_3 be minimum, with given prices. As in Art. 4 then, we obtain the equations

$$\frac{\phi_1(x', z')}{\phi_2(x', z')} = \frac{\theta_1(y', l' - z')}{\theta_2(y', l' - z')} = \mu' = \frac{p_1'}{p_2'} . \tag{7.1}$$

For

$$\delta(x + y) = 0 = \phi_1(x', z')\delta x' + \phi_2(x', z')\delta z$$
$$0 = p_1'\delta x' + p_2'\delta z' \quad ,$$

and, likewise, if $\delta y'$, $\delta z'$ are variations chosen so that $\delta U_3=0$,

$$0 = \theta_1(y', l' - z')\delta y' - \theta_2(y', l' - z')\delta z'$$
$$0 = p_1'\delta y' - p_2'\delta z',$$

$\delta(l'-z')$ being $-\delta z'$, since these virtual changes are made at the time $t=t'$, and l' is given.

By writing (7.1) at the time t (since t' is arbitrary), we have

$$\phi_1(x, z)\theta_2(y, l - z) = \phi_2(x, z)\theta_1(y, l - z)$$

and from (4.1)

$$\phi_1(x, z)\theta_2(\phi(x', z') - x, l - z) = \phi_2(x, z)\theta_1(\phi(x', z') - x, l - z).$$

But this is precisely the first of equations (4.7).

The solution of the problem of Art. 4 thus satisfies the relation, just obtained, which expresses the condition of minimum costs. Moreover, making use of this relation, we may write

$$\int_{a-T}^{b-T} \{\theta_1(y, l - z)\phi_1(x', z') - \theta_1(y', l' - z')\}$$
$$\cdot \left\{\delta x' + \frac{\theta_2(y', l' - z')}{\theta_1(y', l' - z')}\delta z'\right\} dt = \delta \int_a^b U_3(t)dt,$$

so that the vanishing of the left hand member yields again the second of equations (4.7).

8. *Relation to numerical values.*—One may object that as a matter of fact there are no such quantities as U_1, U_3, p_1, p_3, for these are assumed to be composites of different commodities, and each of these commodities must be considered as having a different dimension.[6] These composites therefore require special definitions. In order to make them subject to quantitive analysis, the U_i, p_i must be considered respectively as trade and price indices for groups of commodities or services in the given classes.

What has been given here is evidently an extreme simplification. Nevertheless it admits of interpretation and generalization in various directions. For example, stocks of material are significant in the determination of the motion of an economic system; what treatment can be given them as composite quantities? What happens to the system of Arts. 2 and 5 if p_3U_3 is to be a maximum instead of U_3? The author believes that the mathematics in the present paper is not difficult, and yet that the results obtained with reference to the given hypotheses, particularly the results of Arts. 5 and 7, would not be anticipated without mathematical analysis.

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[6] Evans, *Mathematical Introduction to Economics* (New York, 1930), Ch. II and IX.

DEMAND FUNCTIONS AND UTILITY FUNCTIONS: A CRITICAL EXAMINATION OF THEIR MEANING

By E. H. Phelps Brown

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The statistical derivation of demand functions has now become a familiar part of econometric procedure: passing beyond the rough estimates and unfulfilled projects with which it began, it has been implemented in a large number of statistical studies. In most of this work the functions have been defined in terms of real price; but in the studies of Professors Fisher and Frisch certain methods have also been devised which proceed directly to "the measurement of marginal utility." The critical discussion which this work has occasioned has been concerned in part to ask how far the methods used can yield results of the kind sought, and in part to ask what kind of result it is that should be sought. It is with the second question that this paper is concerned.

The paper begins by giving reasons for believing that the measurement of marginal utility must not be conceived as a measurement of satisfactions; it then considers the formulation of an index of disposition which will embody the essential principle of the marginal utility function without being exposed to the same philosophical objections; and finally it examines the properties of the real price function in comparison with those of the proposed index.

I

Were it possible for economists to build their work upon the doctrine of psychological hedonism, they would have no difficulty in defining an index of the consumer's disposition; but when they are aware of the objections to thinking of man as "a pleasure-machine," the task becomes more difficult. A solution which has gained some acceptance has been to substitute for pleasure or happiness the more colourless term satisfaction, and to say that in speaking of satisfaction the economist offers no doctrine of the determination of human actions, but refers only to the observable fact that in order to acquire certain objects men are willing to make payments: which fact entitles us to consider the degree to which a consumer thinks different objects worth paying for, or, in a word, the payworthiness of different objects to him. When the economist speaks of utility, it is this payworthiness that he should mean. That the consumer does hold an object worth payment, may be explained as the philosopher or psychologist may wish, and to this the economist is indifferent; enough for him to be

able to compare different objects of a consumer's choice in respect of their payworthiness.[1] The same position has sometimes been expressed by means of a distinction between the quality of a desire, and its quantity or force; different actions may be viewed very differently by psychologist or philosopher, but the economist can make quantitative comparison of the *force* of different motives.[2] From such a position it follows that we are entitled to conceive of a quantity of satisfaction, and to define the index of a consumer's disposition towards an object as a function relating the rate of consumption with the quantity of satisfaction derived. Let us now examine the objections to which such a position is exposed.

The statement that men are willing to pay in order to acquire certain objects, is incontestable; but the meaning of the utility or payworthiness of different objects to the consumer is far from clear. It may, in the first place, mean the common quality which commodities have of attracting the consumer; but such a meaning gives us no grounds for conceiving of a quantity of payworthiness; different animals have in common the property of eating, but there is no sense in speaking of a quantity of eatingness. In the second place, however, a reference to the common quality of utility or payworthiness may assert the presence in all the consumer's actions of a common property, the force of his desire; and interpreted in this sense the argument coincides with the second form of explanation set out above, in which we distinguish between the quality of a desire, ethically or psychologically considered, which may be different in different cases, and its force, which is qualitatively the same in all. Now the fact that this form of explanation cannot be expressed without a reference to desire or motive, shows that it involves a departure from neutral territory; and because of that departure, the economist now finds himself faced with the question, whether or not it be the quantity of this force of desire that determines the consumer's action. If he answer no, then he is as-

[1] It is this, I think, that Marshall has in mind, though his form of expression lays him open to the objection that he asserts every human action to arise from the desire for satisfaction: "It is clearly not the part of economics to appear to take a side in ethical controversy: and since there is a general agreement that all incentives to action, in so far as they are conscious desires at all, may without impropriety be spoken of shortly as desires for "satisfaction," it may perhaps be well to use this word instead of "pleasure," when occasion arises for referring to the aims of all desires, whether appertaining to man's higher or lower nature." (*Principles*, 8th edn., I, ii, 1, note.)

[2] "Thus measuring a mental state, as men do in ordinary life, by its motor-force or the incentive which it affords to action, no new difficulty is introduced by the fact that some of the motives of which we have to take account belong to man's higher nature, and others to his lower." (Marshall, *Principles*, 8th edn., I, ii, 1.)

serting that the consumer may choose A although his desire for A is of less force than his desire for B, and this is evidently inconsistent with the proposal to measure the consumer's desire for A from his payments for it. If on the other hand the economist answer yes, then he is asserting that all the consumer's actions are determined by a motive of one and the same kind: for by the force of a desire he now explicitly means its power to prevail and issue in action, and since this force is quantitatively comparable in its several instances, it must in those several instances be one and the same in kind. Here, then, the economist is left under the necessity of asserting that every action of the consumer issues from a motive of the same kind; and this I believe must be his ultimate position in whatever way he try to justify the contention that we can conceive of satisfaction quantitatively. But such a position evidently lies within the realm of philosophy, and is contestable.

It is perhaps worth passing notice, that it is the treatment of all demand as arising from the desire for satisfaction, which compels the economist to introduce the "law of diminishing utility." If it is satisfaction that I seek from all that I buy, then why should I ever buy more things than one? To explain the fact that consumers do buy a diversity of objects, it is then necessary to assume that each commodity yields satisfaction at a diminishing rate. But this "law of diminishing utility," thus conceived, is a very dubious psychological generalization. Resort to it is none the less compulsory for those who hold that "all incentives to action . . . may without impropriety be spoken of shortly as desires for 'satisfaction'."

So far in criticism. But now let us note that the impulse to treat satisfaction quantitatively arises from the fact that in the study of demand we have such abundant quantitative expression of human choices; and nothing that has been said precludes us from using this material to form quantitative indexes of human behaviour: our argument shows only that we must not try to establish a correspondence between quantities in the pricing system and supposed quantities in the realm of thought and feeling. Nor is it only that we may use the data of the price-system to construct the type of index which, like the birth-rate, is descriptive of human behaviour, but makes no attempt to penetrate into motivation: we are at liberty to posit upon the fact of choice a function such, for instance, that any movement by the consumer from one arrangement of consumption to another that he prefers to the first, shall be associated with an increase in the numerical value of the function. At this point we touch on the treatment of the utility function by the mathematical economists; with the findings of the present section in mind, let us go on to examine the scope and significance of this treatment.

II

In the last paragraph it was suggested that the mathematical economist can define a function of utility such, for instance, that for two assortments of goods as between which the consumer is indifferent it shall have the same numerical value, and that if the consumer prefer assortment A to assortment B, it shall have a higher numerical value for A than for B.[3] But the significance of such procedure may be made clearer, and the necessary limitations of the index so defined may be emphasized, if we consider a more elementary statement of the hypothesis upon which it is based.

This hypothesis is, in general terms, that the distribution of a consumer's outlay arises from judgments of equivalence at the margin. If, more particularly, a man is now consuming butter and oranges at certain rates per month, and we offer to give him freely either half a pound of butter or x oranges per month, then we assume that we can by experiment find a value for x such that the consumer is indifferent which form the gift shall take. From this it follows that if we hold his consumption of oranges constant, and vary the rate at which he consumes butter, we can obtain a scale showing how many oranges he considers the equivalent of a marginal increment (half-pound) of butter, at each of a number of different rates of consumption of butter. The process may be repeated with other goods than butter; and we may now further assume (a) that marginal increments which the consumer judges equivalent to the same number of oranges, will be judged by him equivalent to each other; (b) that if the marginal increments are sufficiently small relatively to the existing rate of consumption, an increment which is judged equivalent to n oranges will be judged equivalent to a parcel of two increments each of which is judged equivalent to $n/2$ oranges. But next, we need no longer assume the consumer to be actually obtaining the good which we use as the basis of comparison: the process can in hypothesis be carried through equally well with a good which the consumer does not now know but which he would be glad to have if it were known to him. From this again it is a short step to specify no particular good as our basis of comparison, but to write only for each actual good studied $i_j = f_j(q_j)$.[4]

The generality of this expression must not hide from us the limita-

[3] For examples of the procedure which is here so roughly indicated, see Bowley: *Mathematical Groundwork of Economics*, Introduction; and Frisch, "Sur un Problème d'Économie Pure," *Norsk Matematisk Forenings Skrifter*, Série I, No. 16.

[4] The exposition here given will be recognized as having much in common with that given by Prof. Irving Fisher in his *Mathematical Investigations in the Theory of Prices and Exchange*. Professor Fisher does treat of total utility and consumer's rent, but compare Pt. II, c. iv, "Utility as a Quantity."

tions which are imposed by the implicit reference to a basis-good, and which sharply distinguish our index from any wished-for measure of satisfactions. We are not, for example, entitled to take the unit of the basis-good as standing for a unit of satisfaction, for in that case we should find quantities of satisfaction by integrating our index, and such integration can give only a meaningless result. The index takes such a form as this: that when the consumer is taking butter at the rate of 3 lbs. per month, he holds a marginal increment of butter equivalent to 20 units of our hypothetical basis-good; when the rate is $3\frac{1}{2}$ lbs. per month, to 17 units; and so on. If we then attempt to find what is the total satisfaction derived by the consumer from a consumption of $3\frac{1}{2}$ lbs. of butter per month, we obtain a total made up first of 17 units of the basis-good taken successively inwards from its margin of consumption, and then 20 units taken successively inwards from the same margin (for each comparison is made with respect to one and the same rate of consumption of the basis-good); and so on. But this is a compilation which has no economic significance. While, therefore, we may use our index to infer whether to a given consumer one position is more eligible than another, we cannot use it to measure the satisfaction derived from consumption, or the sacrifice imposed by taxation.

Again, the fact that a consumer's index for a certain good is lower in one period than in another, does not permit us to say that his taste for the good has diminished, or that "its utility is less to him than before." All that we know is that the consumer is willing to give up less of the basis-good than before to obtain a unit of the good studied, and this (to use for once the language of utility) may equally arise from an increase in the utility of the basis-good. In fact the use of such language always goes beyond our evidence: we know only that there has been a change in relative estimation, and we have no unit of estimation common to both periods. If between two periods the index of good A has fallen and that of good B has remained the same, then we can infer that the consumer will give up less of B for the marginal unit of A in the second period than in the first; but we cannot infer that his taste for A has diminished, or his taste for B increased. From this it follows that the significance of our index is in no way impaired if the basis-good be one thing in one period and another in the next. If in one period our basis-good be 100 units of bread, and in the next 50 units of coffee, our index is as significant as if in both periods the basis-good were 100 units of bread: for in either case we have in each period a basis-good which forms a common basis of reference within that period but whose significance within another period cannot be known. This observation will prevent us from considering the rise and fall of our index surface as marking changes in the consumer's taste for the

good studied; and it will also prove to be of importance when we come to consider the real price function.

While in these ways our index stands opposed to the function of marginal utility conceived as a measure of satisfactions, it none the less shares with that function a property of great significance in the analysis of prices. Any function stated in terms of price, whether real price or monetary, is based upon an implicit reference to an existing array of prices: the price which the consumer will give cannot be known without reference to his income and to the several prices set on other goods. It has been the advantage of the marginal utility function that it transcends this limitation, and expresses directly the conditions of demand which are among the determinants of the given array of prices, but which equally are consistent with other arrays. This property our index shares. It rests directly upon the consumer's judgment of equivalence, and is expressed in terms of the number of units of the basis-good which, at the margin, he deems equivalent to a marginal increment of the good studied: and these terms are not dependent on the prices ruling in the market or on the income of the consumer.[5]

An examination of statistical work employing the concept of marginal utility will show that the procedure in which the function of marginal utility is employed remains equally possible if by that function we understand throughout an index of the kind here described. Philosophical critics are therefore too hasty if they dismiss such work on account of the implications either of "measurement" or of "utility." But statisticians for their part must recognize the limitations of their procedure, and restrain it from all applications which involve the concept of a quantity of satisfaction.

III

In the light of the foregoing considerations, we have now to examine the significance of the real price function, in which as yet most statistical studies of demand have been expressed.

The real price function can be conceived in a form closely analogous to the index as here defined, but in one essential respect the analogy is incomplete. The likeness appears in that we may take as our basis-good the total real income of the consumer or consumer-group, and real

[5] I here assume that the consumer's attitude towards each of the goods studied is independent both of the several amounts of other goods that he may be obtaining and of such amounts in conjunction, i.e., of his real income. This assumption is arbitrary; but it greatly simplifies the exposition, and its removal does not vitiate the conception of an index of disposition which the exposition is designed to establish.

price then appears as a quantity of the basis-good, the quantity namely which is deemed equivalent to the marginal unit of the good studied. We have seen moreover that the significance of our index is not affected by changes from period to period in the basis-good, so that the real price function will not be vitiated by similar changes in the amount of real income. The real price function therefore might be considered as a particular case of the general form of our index, were it not for a difficulty which has already been mentioned, a difficulty to which price functions are exposed but from which our index is free. This is as follows.

Any price function can be valid only so long as it be the sole such function considered at the time: for each price function represents the changes in price that will be associated with changes in the rate of consumption of a good, when the price of every other good is held constant.[6] We therefore cannot consider two such functions at the same time, for each supposes the constancy of a price which the other supposes to vary.

It is the neglect of this limitation which vitiates Marshall's argument on the redistribution of output between industries of increasing and decreasing return (*Principles*, 8th. edn. v, xiii, 4); this argument would hold only if, apart from many other assumptions, we could assume that the price of each product could change without altering the price demand function for the other. That this assumption leads to a wrong conclusion is shown by the following alternative treatment, in which no use is made of a price function of demand. Let us suppose that in the position of market equilibrium the community is buying i units of the good of increasing returns, I, and d units of the good of decreasing returns, D; and let the point (i, d) be inscribed in a graph where I and D are measured by equal intervals along the horizontal and vertical axes respectively. Then the indifference curve passing through (i, d) will be convex to the origin. Now the form of the transformation curve which passes through (i, d), and which shows what output of either good is made possible by a given reduction in the output of the other, is indeterminate: for while the relinquishing of successive equal increments of D sets free successively smaller increments of productive resources, the application of successively smaller increments of resources can yield successive equal increments of I. But if we assume that (i, d) is a point of stable equilibrium, we exclude the possibilities

[6] Even this definition of the price function can hold only for the *impact* of price or quantity changes, and not for the resulting equilibrium: for if the demand for the good studied be not of unitary elasticity, a change in its price will cause changes in the quantity of money expended on other goods, with consequent effects upon their prices.

that in the region of (i, d) the course of the transformation curve lies beyond or on the indifference curve. We are then left with the possibility that in the region of (i, d) the transformation curve yields points which lie between the indifference curve and the origin. But any such point records an arrangement of consumption less eligible than that recorded by (i, d).—Marshall's argument, however, escapes this refutation, if we understand it to rest upon the possibility that the transformation curve which applies to a *sustained* change in demand is different from that which applies to the *impact* of an experimental and unsustained change.

We are faced, then, with a seeming paradox. On the one hand the real price function may be considered as a case of our index; on the other hand, any price function refers to a particular context of prices, from which it cannot be separated without risk of self-contradiction. But the explanation of the paradox lies in the definitions of real income and real price. Since there will not in general be a homogeneous unit of real income or real price, these concepts are usually defined only to be equal to the quotient of a figure of money divided by a price-index. Now as we suppose variations in the price of the good studied, we must in theory thereby imply variations in the price-index, and with these, changes in the basis-good and the unit of real price. Not only, moreover, does the basis thus change as we move from point to point in the demand function for one good, but it changes in different ways in the demand functions of different goods: variations in the price of butter will affect the basis in one way, variations in the price of bread will affect it in another. No two real price functions can therefore be strictly comparable, that is, capable of being considered simultaneously.

Thus the real price function comes near to being a case of the index of disposition, but is marked off from that index by a limitation which is of potential importance: even though for statistical purposes it may sometimes be safely neglected. While philosophical considerations make our index of disposition preferable to the utility function, the index of disposition shares with the utility function an advantage in economic analysis which makes it preferable to the price function.

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THE REACTION OF CONSUMERS TO CHANGES IN PRICES AND INCOME

A QUANTITATIVE STUDY IN IMMIGRANTS' BEHAVIOR[1]

By HANS STAEHLE

I. THE PROBLEM

CONSIDER a given occupational and social group, for instance, industrial workers. If prices or the income of the group change, the composition of their food budgets will change in a characteristic way. The nature of this change is important from several viewpoints, not least in connection with the analysis of demand schedules, utility curves, and similar econometric studies.

An interesting possibility of studying this variation seems to be offered by the information that is at present available on the behavior of *immigrants* into the United States. In this paper we shall in particular consider Scandinavian, and especially Swedish, immigrants. A comparison between the food budgets of working men's families in Sweden with the food budgets of the Swedish immigrants after their arrival in the United States, and also with the food budgets of the corresponding class of native American (as defined later), ought to furnish interesting results.

The Americans and Swedes are living in a country where the economic conditions, and especially the prices, are in harmony with their habits and tastes; the Scandinavian immigrants, depending in varying ways on the degree of their assimilation, or "Americanization," are in all cases more or less in conflict with the economic conditions of the country they live in: their tastes and habits are more or less Scandinavian, and predominantly Swedish, but the prices they have to pay for their food are American, and things which are agreeable to Swedish palates may be relatively more expensive and others less agreeable may be cheap, in the American system of prices. In these circumstances what has been the behavior of the Scandinavian immigrants? This is the question to which an answer will be sought in the following sections.

The authors of the British Board of Trade Report on cost of living in American towns[2] were perfectly aware of this problem and dealt

[1] The author wishes to express his gratitude to the Rockefeller Foundation, which supplied the financial means for this study, and to Mr. A. Goldberg, without whose never failing enthusiasm the following computations could not have been carried out.

[2] *Cost of Living in American Towns, Report of an Enquiry by the Board of Trade*, London 1911 (Cd. 5609). The following page references relate to this publication.

with it to some extent. They remark that "national habits and practices as regards the choice of food are very tenacious of life, even when transferred to an entirely new country, and for some peoples in the United States the old home and its ways are kept in memory by fresh arrivals who tend to seek their compatriots in the new land. Language also, in spite of the schools and their effect upon the rising generations, is often a bond that tends to maintain the persistence of the national life . . . (p. xxxix). Although, however, differences exist in this composite national life that tend to blur the picture of the American type, . . . it is nevertheless necessary to draw attention to the fact that even in relation to the alien peoples of the United States 'American' speedily comes to have a meaning of its own . . . and the 'Americanization' of immigrants is apt to begin almost from the moment of their landing (p. xl). A mental attitude has been created that in an indefinable, but not less real way, appears to minimize even ethnological differences and to have resulted in an outlook that is industrially and even temperamentally 'American' (p. xiv). . . . It must, however, be borne in mind that many stages in the process of assimilation, both of the first and of succeeding generations of immigrants, to a national type are represented, and that thus each ethnological group is itself more or less a composite. Moreover, it cannot be assumed that as regards the stages of assimilation the races represented are in their true relative proportion either in respect of the United States as a whole or of the towns investigated" (p. xxxix–xl).

These remarks may be summarized as follows: whereas national habits, especially as regards the choice of food, are tenacious of life with the immigrants, there exists clearly some true American type which, though itself the result of the contributions of various races, tends to become the ideal, if not of the first generation of immigrants, yet certainly of their children. But this is the case in different degrees for different individuals and for different races; as a result, the information collected on the immigrants, though classified by national origin, may be heterogeneous, depending on different degrees of assimilation to the American type.

The present study may be understood to be a sort of continuation of the above observations in that it attempts to find out whether nothing more generally true can be said, by use of essentially the same material, on the behavior of a particular racial group, i.e., the Scandinavians, with respect to the problem of Americanization.

II. THE MATERIAL

The material used for the following analysis consists of: (1) the quantities of 19 foodstuffs consumed, and the prices paid for them,

by American workingmen's families, classified according to the income per family in 1909; (2) the quantities of the same foodstuffs consumed, and the prices paid for them, by immigrant workingmen's families of Scandinavian origin living in the United States in 1909, and classified according to the income per family; (3) the quantities consumed of 17 foodstuffs, and the prices paid for them, by Swedish workingmen's families living in Sweden in 1913–14 classified according to the income per consumption unit ("equivalent adult male").

These three sets of data are reproduced in Table I, where further particulars are given in the notes. Only one series of prices has been given both for the Americans and the Scandinavian immigrants. The differences in the average prices paid are in fact so small that it did not seem worth while to insist on them. On the other hand, the adoption of different prices for these two groups would have involved a very considerable increase in the amount of computation work.

The American figures were collected by the British Board of Trade during February, 1909.[3] They relate exclusively to industrial workers' families. "Owing to the multiplicity of races represented in the budgets of family income and expenditure on food and rent," it was found expedient "to represent the results on a nationality basis, according to the declared country of birth of the head of the family. . ." (p. iv) "although in the case of mixed marriages the importance of the wife in the *ménage* is often predominant" (p. xxxix). On this basis 10 different nationality groups of budgets, each classified by the income per family were formed. Some particulars may be given relating to those which have been used in this paper, namely the "American-British (Northern)" and the Scandinavian groups.

As regards the first, which the Board of Trade considered as representing the characteristics of the "true" American people; this covers 3215 families living in 22 "northern" towns, i.e., New York, Boston, Brockton, Fall River, Lawrence, Lowell, Providence, Baltimore, Newark, Paterson, Philadelphia, Cincinnati, Cleveland, Detroit, Louisville, Muncie, Pittsburgh, Chicago, Duluth, Milwaukee, Minneapolis-St. Paul, and St. Louis. The group is made up of 2278 American-born families, 436 Irish, 227 English and Welsh, 189 Canadian (mostly of French descent), and 85 Scottish. These indications relate to the group as a whole. They are given in the absence of particulars concerning the 2637 families (having incomes below £6 per week), data for which have been used exclusively in the present paper.

As regards the Scandinavian immigrants, the total number of families amounts to 335, of which 231 were Swedish, 96 Norwegian and 8

[3] *Loc. cit.*

TABLE I. AVERAGE ANNUAL QUANTITIES OF FOODSTUFFS CONSUMED AND FAMILIES IN THE UNITED STATES

Articles	American families in 1909					Scandinavian U. S. A.	
	Yearly consumption per family consisting of:					Yearly con con	
	3.78 persons	4.08 persons	4.54 persons	5.02 persons	5.27 persons	4.09 persons	4.60 persons
	and having yearly incomes per person of:					and having	
	$105	$160	$196	$223	$258	$161	$191
1. Wheaten bread	118.4	154.0	180.2	206.2	214.4	77.8	63.2
2. Rye bread	15.3	22.6	20.5	17.5	20.0	43.9	56.1
3. Wheaten flour	224.6	187.3	212.0	247.9	277.6	209.9	270.5
4. Rye flour	—	0.9	1.7	1.4	2.1	37.5	33.7
5. Oats, meal, & flour	18.2	22.6	29.0	33.0	34.9	33.7	44.8
6. Potatoes, Irish	370.1	411.1	438.5	499.6	542.3	323.4	426.9
7. Beef	84.7	120.1	142.5	158.3	184.2	122.7	124.1
8. Mutton	9.2	16.3	21.5	29.0	34.9	25.7	34.7
9. Pork	36.6	45.8	50.7	51.2	52.8	46.2	61.6
10. Bacon, ham, etc.	24.5	29.7	34.4	43.2	42.7	22.4	28.1
11. Veal	9.0	10.9	18.9	21.5	23.6	11.1	15.3
12. Sausage	6.4	12.0	16.3	17.7	19.3	17.5	19.1
13. Lard	25.5	27.4	30.4	34.9	36.3	19.8	24.5
14. Butter	26.9	31.8	41.0	50.7	55.7	42.5	55.7
15. Cheese	5.7	7.3	10.6	13.2	14.2	10.4	14.4
16. Fresh milk	175.2	221.5	281.8	322.9	350.0	350.0	450.3
17. Eggs	469.6	753.5	1034.8	1252.7	1317.7	924.0	1096.2
18. Coffee	14.9	18.2	21.9	23.4	25.7	21.9	27.1
19. Sugar	84.0	89.2	105.0	133.7	137.0	106.1	121.2

N.B.: The quantities are given *in kilogrammes* except for milk (no. 16) which is given in litres, and for eggs (no. 17) which are given in units.

SOURCES.—*American Families in 1909: Cost of Living in American Towns, Report of an Enquiry by the Board of Trade*, London, 1911 (Cd. 5609), pp. xlvi and l. The quantities consumed there given in lbs. per week as collected during February, 1909, have been converted into kilogrammes per year. The prices have been obtained by dividing expenditure by quantities. They are weighted according to the number of families in each income group. The conversion from English into American currency has been carried out at par.

Scandinavian immigrants living in U. S. A. in 1909: Cost of Living in American Towns, as above quoted, pp. 412 and 413. The quantities given for the first four income groups have been obtained by the same procedure as described for the American families. The period of observation is also the same. No special set of prices is given here, since the prices paid by the immigrants are practically identical with those paid by the Americans. The quantities, the income, and the

AVERAGE PRICES PAID FOR THEM BY CERTAIN GROUPS OF WORKINGMEN'S (1909) AND IN SWEDEN (1914)

immigrants living in in 1909			Swedish families in 1914				Average prices paid per kilogramme:	
sumption per family sisting of:			Yearly consumption per unit in families consisting of:					
5.15 persons	6.02 persons	6.06 persons	5.57 persons	4.34 persons	3.37 persons	2.75 persons	in U. S. A. 1909	in Sweden 1914
yearly incomes per person of:			and having yearly incomes per person of:					
$217	$225	$308	288.3 kr. =$77	393.4 kr. =$105	533.4 kr. = $143	841.5 kr. = $226	cents	crowns
77.1	67.2	80.2	6.3	6.6	7.2	6.3	12.21	0.55
60.1	15.8	67.5	82.2	73.8	73.8	64.3	11.16	0.31
254.0	379.0	366.2	49.6	53.1	49.4	43.5	8.46	0.31
45.3	71.0	56.7	25.8	21.8	16.0	13 4	7.34	0.25
41.3	42.5	43.0	11.7	12.1	11.4	13.7	15.00	0.41
536.1	654.1	594.4	126.1	132.5	127.1	128.3	4.00	0.067
123.8	147.7	175.8	9.6	10.8	12.6	13.9	34.30	1.15
36.8	47.9	47.3	0.9	1.1	1.3	2.7	35.85	1.27
53.5	83.3	74.9	9.2	10.7	11.5	13.0	32.25	1.26
30.4	37.7	38.4	4.6	5.3	5.6	5.5	38.30	1.41
41.7	30.0	31.1	2.9	3.9	4.9	5.0	34.35	1.00
20.5	17.9	18.5	—	—	—	—	27.54	—
24.1	28.8	36.0	—	—	—	—	30.16	—
69.8	82.1	82.3	9.8	12.4	14.4	17.7	68.00	2.18
18.9	16.7	21.5	4.6	5.4	6.1	6.6	42.70	1.45
473.7	567.8	570.2	189.7	251.9	210.8	347.5	7.45	0.142
1254.2	1524.1	1689.1	120.0	170.0	209.0	269.0	2.28	0.067
30.4	32.6	33.5	6.3	6.5	7.4	7.2	53.07	2.24
126.9	159.0	193.6	29.0	32.6	34.8	39.4	12.74	0.63

number of persons in the family above given for the fifth income group, have been computed as unweighted averages of the figures collected for families having incomes of more than about $24 per week. The fourth above group is included in this average, and the weight attached to it is one fourth, the average being one of four income groups. The procedure is evidently not very satisfactory, but owing to the small numbers of families included in each of the higher income classes, no other device was applicable, if the upper levels of income were to be considered at all.

Swedish families in 1914: Levnadskostnaderna i Sverige 1913–1914, Del I. Utredningens Huvudresultat, av K. Socialstyrelsen, Stockholm, 1921, pp. 32–35 and 72–73. The quantities there given per unit of consumption (equivalent adult male) per year have been given as published, except for potatoes, which are published in litres and were converted into kilogrammes. The prices have been obtained by dividing expenditure by quantities and weighting by the number of families in each income group.

Danish, living in 16 towns. 84 Swedish and 51 Norwegian budgets were obtained from Minneapolis-St. Paul. Chicago, New York, Duluth, Boston, and Providence, provided all others, with the exception of 81.

The data relating to Swedish families living in Sweden were collected by the Swedish Social Board mainly during the year 1914.[4] 1355 budgets were included, but only data concerning workers' families, numbering 933, have been used in the calculations. Of these, 800 were living in 36 cities and industrial towns of the more southern parts of Sweden (south of 60° latitude north), excluding Stockholm, but including Uppsala, Jönköping, Malmö, Hälsingborg, and Göteborg. The remaining 133 were distributed over Norrland, from Gävle to Luleå and Kiruna.

III. THE TOOL

We shall in particular attempt to measure the degree of similarity or dissimilarity between the various food budgets. The formula which we adopt for this purpose is based on the deviations of the "quantity ratios" for corresponding foodstuffs consumed in countries "I" and "O" from the weighted average of these quantity ratios. Writing q_1 and q_0 for the quantites of foodstuffs consumed, and p_1 and p_0 for the prices paid for these foodstuffs in countries "I" and "O" respectively, each deviation above mentioned may be put as follows:

$$(1) \qquad \left(\frac{q_1}{q_0}\right) - \frac{\Sigma\left(\frac{q_1}{q_0}\right) q_0 p_0}{\Sigma q_0 p_0} = \left(\frac{q_1}{q_0}\right) - \frac{\Sigma q_1 p_0}{\Sigma q_0 p_0}.$$

The weighted average of these deviations, taken without respect to sign, and weighting by the expenditure in country "O" (namely $q_0 p_0$) then becomes:

$$(2) \qquad \Sigma \left| q_1 p_0 - q_0 p_0 \frac{\Sigma q_1 p_0}{\Sigma q_0 p_0} \right| \frac{1}{\Sigma q_0 p_0},$$

and dividing (2) by the weighted average of the quantity ratios (i.e., $\Sigma q_1 p_0 / \Sigma q_0 p_0$) the following expression is arrived at:

$$(3) \qquad \Sigma \left| \frac{q_1 p_0}{\Sigma q_1 p_0} - \frac{q_0 p_0}{\Sigma q_0 p_0} \right| = D.$$

This is a measure of dissimilarity which may have values varying be-

[4] *Levnadskostnaderna i Sverige 1913–1914*, Del I. Utredningens Huvudresultat, av K. Socialstyrelsen. Stockholm, 1921.

tween 0, indicating the greatest possible similarity, and 2, indicating the greatest possible dissimilarity.[5]

It has been found that this measure is highly sensitive and shows considerable variations when the consumption of families belonging to different income classes is considered in comparison with any income class of the same or another country.[6] In particular, when the food consumption (or part of it) pertaining to each one out of a range of incomes in country "I" is compared with the food consumption of each out of a whole range of incomes in country "O," by computing D for all possible combinations, the surface thus obtained exhibits more or less clearly a sort of valley; and it is possible to conclude from this "valley", by a number of appropriate hypotheses, that the incomes in countries "I" and "O" corresponding to the two sets of data on food consumption which have yielded the minimum (or "valley") values of D are reciprocally equivalent. The characteristic valley shape holds true, however, only in the case where there is a complete adjustment between prices and consumption habits. It will be seen that, e.g., in the case of immigrants, where this adjustment is not complete, the variation of the values of D is subject to other influences and therefore different in character.

IV. THE RESULTS

Tables II and III give the result of computing the values of D for all possible combinations of the sets of data shown in Table I. Table II compares the Scandinavian immigrants, arranged into five classes according to family income, with the Americans, classified into five groups according to family income; since only one set of prices was adopted for both groups, D and D' have the same values; 19 foodstuffs have been considered. Table III compares the same Scandinavian immigrants with the Swedes living in Sweden; in this case Sweden has been numbered "O" in formula (3); 17 foodstuffs are included in the computation. Both Tables II and III, with the exception of the last column that is dealt with below under 2*d*, are represented graphically in Figures 1 and 2. These diagrams should, however, be examined only

[5] By taking the deviations (q_0/q_1) instead of (q_1/q_0) and weighting by q_1p_1 instead of q_0p_0, an alternative measure of dissimilarity may be written as follows:

$$(4) \qquad \Sigma \left| \frac{q_0p_1}{\Sigma q_0p_1} - \frac{q_1p_1}{\Sigma q_1p_1} \right| = D'.$$

But the movements of D and D' are generally too similar to make the double computation worth while. It will be specified in each case later on which country is taken as "I" and which one as "O," in terms of expression (3).

[6] See the author's "Ein Verfahren zur Ermittlung gleichwertiger Einkommen in verschiedemen Ländern" in *Archiv für Sozialwissenschaft und Sozialpolitik*, June 1932, pp. 436–446.

TABLE II AND III. VALUES OF THE MEASURES OF DISSIMILARITY (D) OBTAINED IN THE COMPARISON OF FIVE INCOME GROUPS OF SCANDINAVIAN IMMIGRANTS WITH FIVE AMERICAN INCOME GROUPS (TABLE II) AND FOUR SWEDISH INCOME GROUPS (TABLE III) TABLE II

Groups of American families having annual incomes per person of:	Groups of Scandinavian Immigrants having annual incomes per person of:				
	$161 (1)	$191 (2)	$217 (3)	$225 (4)	$308 (5)
$105	0.2983	0.3295	0.3993	0.3433	0.3255
$160	0.2296	0.2835	0.3411	0.3174	0.2964
$196	0.1951	0.2616	0.2780	0.2797	0.2485
$223	0.1901	0.2614	0.2714	0.2704	0.2306
$258	0.1888	0.2552	0.2669	0.2687	0.2347
Groups of Swedish families having annual incomes per person of:	TABLE III				
	(1)	(2)	(3)	(4)	(5)
Kr. 288.3 or $77	0.5398	0.5153	0.5246	0.5594	0.5335
Kr. 393.4 or $105	0.5088	0.4724	0.4728	0.5135	0.4878
Kr. 533.4 or $143	0.5064	0.4568	0.4457	0.4864	0.4629
Kr. 841.5 or $226	0.5132	0.4677	0.4357	0.4758	0.4578

with reference to the corresponding tables. But to make the reasoning easier, it runs in terms of the graphs wherever possible.

The following features may be noted:

1. The average heights of the two bodies which are made similar in the diagrams—or the average level of dissimilarity in the combinations of food composites they represent—are in fact quite different. In Figure 1 the values of D vary from 0.189 to 0.399; they go from 0.436 to 0.559 in Figure 2. And when a comparison is made between

TABLE IV.—VALUES OF THE MEASURE OF DISSIMILARITY (D) OBTAINED IN THE COMPARISON OF FIVE INCOME GROUPS OF AMERICAN FAMILIES WITH FOUR INCOME GROUPS OF SWEDISH FAMILIES

Groups of Swedish families having annual incomes per person of:	Groups of American families having annual incomes per person of:				
	$105	$160	$196	$223	$258
Kr. 288.3 or $77	0.6743	0.6794	0.6891	0.6935	0.7013
Kr. 393.4 or $105	0.6687	0.6768	0.6645	0.6588	0.6655
Kr. 533.4 or $143	0.6798	0.6868	0.6612	0.6523	0.6548
Kr. 841.5 or $226	0.7143	0.7088	0.6862	0.6709	0.6754

the true Americans and the Swedes living in Sweden, the values for D fluctuate between 0.652 and 0.714, as shown in Table IV. The smaller

range in this latter case, as compared with that found in Tables II and III, suggests that the variation in the values of D in the Table IV is due mainly to the variation in the incomes considered, and that at any rate no such factors as lack of adjustment between habits and prices intervene which apparently have a violent effect upon the degrees of

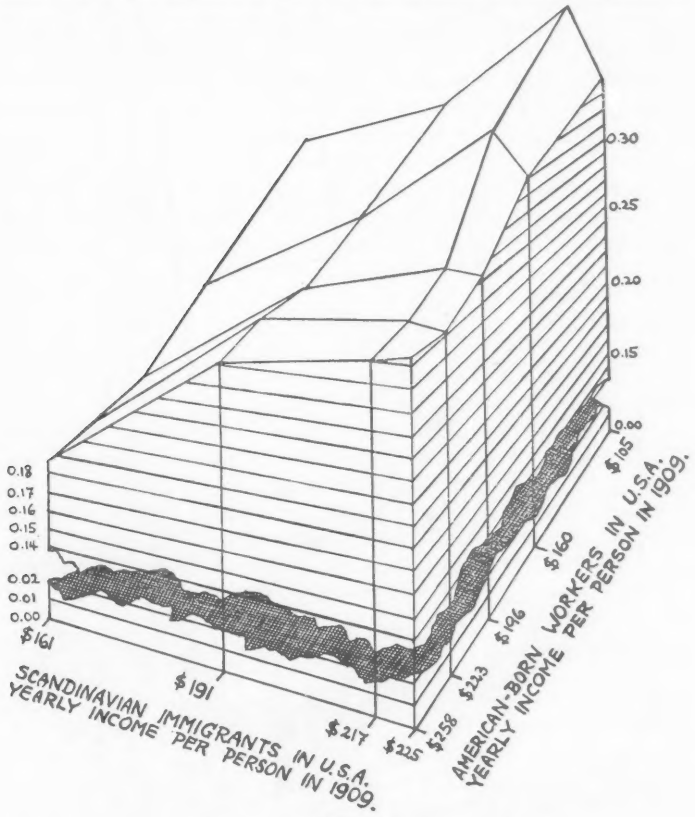

FIGURE 1

dissimilarity. In other words, the food consumption of Scandinavian immigrants is, generally speaking, more similar to that of true Americans than to the food consumption in the home country; it shows, however, greater similarity to the consumption of the Swedes than does the consumption of the Americans.

2. The variation of the values of D from left to right, apart from

their level, is also very different in the two diagrams: (a) The food consumption of the immigrants belonging to the lowest income group

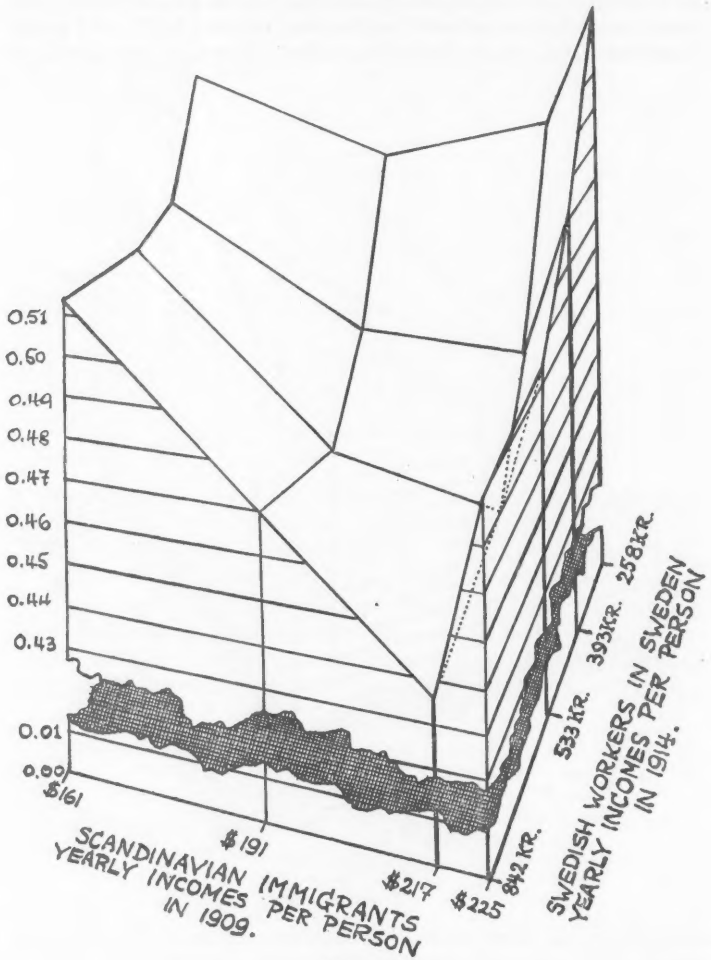

FIGURE 2

considered appears generally to be more similar to the consumption of any of the five American income groups than is that of the second and third immigrants' income classes, as indicated by the increasing

values of D (heights in Figure 1) from left to right. (b) On the other hand, the food consumption of the lowest immigrants' income group is generally less similar to the consumption of any of the Swedish income groups than is that of the second and third immigrants' income classes, as indicated by the decreasing values of D (heights in Figure 2) from left to right. (c) These two contrary movements of the dissimilarities found for the first three immigrants' income groups are reversed when the fourth group is taken into consideration. In Figure 1, where dissimilarity is increasing from the first to the third class, it suddenly drops, and in Figure 2, where it had been decreasing, it suddenly rises, for the fourth income class of the immigrants. (d) When, however, the fifth immigrants' income class (which is not shown in the diagrams, but only in Tables II and III) is considered, it is found that the movement which was in opposite directions in both tables for the lower four immigrants' groups, becomes parallel from the fourth to the fifth class: a drop is shown in both Tables II and III, on all income levels taken into consideration.

3. Looking now at the figures shown in Tables II and III and reading the columns from the top to the bottom (rather than at the diagrams where, owing to the perspective, the movement is less clear), it is seen that: (a) The food consumption of any given immigrants' income class invariably becomes more similar to the Americans' consumption, the higher the American income class considered, as shown by the decreasing values of D in the vertical columns in Table II. But the movement is not identical for all four immigrants' groups,[7] and the decrease of D is strongest for the lowest of the four classes. (b) The food consumption of any given immigrants' income, when compared with that pertaining to increasing incomes in Sweden, shows variations of a less regular nature. Reading the vertical columns in Table III, minimum values for D are found for the lowest and the second immigrants' groups (cols. (1) and (2)) with respect to the third Swedish class; the minima for the third and fourth immigrants' groups remain undetermined, but it may be assumed that for the third class it would lie opposite the fourth, or between the third and the fourth Swedish classes, whereas for the fourth immigrants' class it would fall either opposite or beyond the fourth Swedish income group. Thus, it is only in the comparison of the immigrants with the Swedes, but not with the Americans, that traces of the usual movements of the dissimilarities are found. And it would appear, though the movement is not reciprocal (i.e., what is a minimum reading the columns from the top to the bottom is not also a minimum reading the arrays from the left to the right in Table II)

[7] The fifth, which is rather artificially obtained by averaging, as explained in the notes to Table I, is left out of consideration in this connection.

that annual incomes per person of Scandinavians in the United States of $161–191, and of $217–225 in 1909 were roughly equivalent to annual incomes per person of 533 crowns (i.e., $143 at par) and 842 crowns (i.e., $225 at par) of Swedes in "the old country" in 1914.

V. THE RESULTS INTERPRETED

The above results may be explained as follows:

First, the fact that the consumption of the Scandinavian immigrants generally becomes *less similar* to that of any American consumption considered when their income increases from the first to the second and from the second to the third class (see above 2a), taken in conjunction with the fact that the consumption of these same immigrants becomes *more similar* to that of any Swede in Sweden considered when their income increases from the first to the second and then to the third class (see above 2b), would indicate that the tastes of the people belonging to these first three immigrants' groups are still *fundamentally Swedish*. For, whereas the conditions of the American market, and notably the price system, impose upon the poorest immigrants (those belonging to the lowest income class) a food consumption which is rather similar to that of Americans, those belonging to the second and third class take advantage of their higher incomes to fall back into the old home habits, and their food therefore becomes more nearly Swedish.

Second, the situation is reversed when the fourth Scandinavian immigrants' class is considered. Food consumption which was becoming increasingly Swedish with rising income suddenly becomes less Swedish and more American (see above 2c); and it would therefore appear that tastes in this case are becoming *predominantly American*, assimilation begins to make itself more strongly felt. It is not, however, clear what the reasons for this may be. One might assume, e.g., that there probably exists a strong interrelation between assimilation and economic success, where assimilation might just as well be an essential condition (if not the cause) as a result of success in economic life. Or else it might be supposed that there is a strong correlation between the period elapsed since arrival in the new country on the one hand and both assimilation and higher income on the other. Or, finally, and this looks most probable, the greater assimilation might be ascribed to the growing economic influence of the (probably American-born) children in the family. In fact, their contribution to the family income amounts to quite insignificant sums in the two lowest classes, but reaches about 12 per cent in the third, exceeds 15 per cent in the fourth, and finally (in income classes which have not been considered individually in the present study) amounts to about 43 per cent of the

total. Moreover, there is no doubt that the influence exerted by the children would be in the direction of "Americanization"; for, as the Board of Trade Report notes (p. 271), "the Germans and Scandinavians (living at Minneapolis and St. Paul, where a great number of Scandinavian budgets were obtained) are markedly frugal and thrifty, and those of the second generation are found in all ranks of business and professional life."

Third, as regards the variation of the dissimilarities obtaining, when a given immigrants' group is compared successively to increasing American incomes on the one hand, and to increasing Swedish incomes on the other, two tendencies are manifesting themselves: that making for the variations already dealt with, which depend on the degree of Americanization (and indirectly on income), and the other usually met with in international comparisons making for a "valley" in the dissimilarity surface. Now, the former influence predominates, as is natural in a comparison of "unadjusted" immigrants with other "adjusted" populations, and this more so in comparing the immigrants with the Americans than with the Swedes. This is why the values of D constantly decrease in the vertical columns of Table II, but move less regularly in those of Table III. There is only one fact which may appear surprising: the poorest immigrants are precisely those whose food consumption is most similar to that of Americans belonging to the highest income class considered (see above 3a), whereas it is relatively more similar to that of the third Swedish income group (see above 3b). These circumstances may, however, be explained by the fact that these poorest immigrants are probably also the most recent ones; and though they are compelled by the pressure of market conditions to adopt the American way of eating more than immigrants belonging to the two higher classes, it is mainly for lack of experience that, instead of consuming in a way as similar as possible to, and therefore as economical as, that of equally poor Americans, they choose as a standard the consumption habitual to Swedes whose income at home they consider to be approximately equivalent to their own in the foreign country.

Thus, our analysis permits of concluding that, in spite of the variety of degrees of assimilation which the Board of Trade reported regarding the immigrants to the United States, at least for the Scandinavians a definite distinction may be made between those who are, as concerns their food bill, still predominantly under the influence of their country of origin, and others who are already largely Americanized. And it appears that the border line coincides with a certain level of income. Moreover, it is remarkable that, below this level, increased income is used to cultivate "Swedishness" and that the variations from one grade to the other are similar to those which are found as between

corresponding grades in Sweden (see above 3b). Finally, it is worth noting, and strongly supports the conclusions of this analysis, that the movement of the dissimilarities above this level (i.e., from the fourth class of immigrants upwards), is such as to make the presence of normally "international" variations most probable, although the presence of a "valley" cannot be checked in detail (see above 2d).

VI. CONCLUSIONS

It is, of course, in the nature of things that there is no way of deciding by the measure of dissimilarity *why* the facts are such as measurement would show them to be; but it is notable that these facts can easily be explained by supplementary and non-quantitative reasoning. By the latter alone it would, however, not have been possible to arrive at the results shown in the preceding sections.

The present writer firmly believes that it is possible, in economics, in sociology, as in other fields, to arrive, by the development of appropriate tools, at results which would be of a more reliable kind than reasonings which have as their most solid, and often as their only, foundation the eloquence of their authors.

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THEORETICAL STUDIES OF DEMAND

By Charles Frederick Roos

A paper read before the joint meeting of the Econometric Society, the American Society of Mechanical Engineers, the American Institute of Electrical Engineers and the American Society for Testing Materials, Chicago, June 30, 1933.

I. THE RÔLE OF MATHEMATICS IN ECONOMICS

So many mathematical economists—Cournot, Walras, Pareto, Fisher, Frisch, Evans, Schultz, and others,—have already given such excellent reasons for employing mathematics in economics that it seems unnecessary for me to add anything. However, in order that my own viewpoint may be made clear, I shall briefly sketch some of the things that I believe can be accomplished by employing the mathematical method.

Suppose that an engineer desired to design a shaft to carry a load of, say, ten tons. He would probably begin by assuming the modulus of rupture of the material selected to be approximately the same as the average experimental modulus of rupture of similar material. By means of mathematics familiar to engineers, he would then be able to calculate the dimension of the shaft. However, he would use a shaft of larger dimension to allow for a factor of safety—his material might not be as strong as the average material, the load might become slightly more than ten tons, and so forth.

As an entirely analogous problem one might propose to determine what the price of a piece of machinery should be in order that one thousand (or one million) units would sell within a year after production, but someone would almost surely ask how changes in advertising and sales activity, tariffs, currency changes, taxes, and other elements, would be taken into account. The obvious reply is that when an engineer calculates the size of a shaft to carry a specified load, he assumes such things as that the bearings will be properly oiled, and that no sand or grit will be thrown into them. If he allows his bearings to heat excessively, or to be subject to undue wear, he recognizes that he has complicated his problem. Similarly, a mathematical economist recognizes that the problem of calculating demand when advertising, sales activity, and other factors vary, is more complicated than the one for which these are assumed to be constant. In order to solve the complicated problem the mathematical economist will, of course, need to know the laws of variation of price and demand with advertising outlay, etc.

Some economists and others have said that there are so many variables involved in a study of human behavior that it will never be possible to develop a science of economics. To these who would use this as an argument for not using mathematics in economics, one might reply that because there are so many variables, there is all the more need for an exact language to keep track of them.

It is true that individuals seem to react so differently to situations that it may never be possible to predict how a given individual will react to a particular situation. For example, in the consummation of a sale, such elusive factors as the personality of the salesman, individual prejudices, misinformation or, perhaps more often, lack of information, play such important rôles that it may never be possible to determine the demand of any particular individual for a given product.

However, although individuals often appear to react capriciously, nevertheless, for a large enough group there ought to be what might be called an average salesman, and an average purchaser. These average individuals might be expected to react almost identically to identical situations, for otherwise there could be no norm of rationality. Thus, Professor Ragnar Frisch, in his book on measuring marginal utility, postulates an average person by using data relating to a group to measure the flexibility of the marginal utility of money.[1] Again, although it may not be possible to predict how many telephone calls John Smith will make at five or ten cents per call, it may be possible to predict how many calls 100,000 John Smiths will make.

From this point of view, mathematical economics is not unlike what mathematical physics *is becoming*. For example, the following quotation appeared in the July, 1927, number of the journal *Engineering:* "Today the mathematical physicist seems to be more and more inclined to the opinion that each of the so-called laws of nature is essentially statistical, and that all our equations and theories can do is to provide us with a series of orbits of varying probabilities."

In 1838 Augustin Cournot succeeded in driving the entering wedge of mathematics into economics when he published his *Researches into the Mathematical Principles of the Theory of Wealth.*

Although Cournot wrote almost a century ago many of his ideas seem to be quite modern. For example, his words "It would seem proper first to investigate the causes which produce absolute variations in the value of the monetary metals, and when these are accounted for, to reduce to the corrected value of money the variations which occur in the value of other articles" have been echoed and paraphrased

[1] Ragnar Frisch, *New Methods of Measuring Marginal Utility* (Tubingen: Verlag von J. C. B. Mohr (Paul Siebeck), 1932). See also the review by professor Henry Schultz, *Jour. of Pol. Econ.*, Vol. 41, Feb. 1933.

many times by modern economists. Practically everyone realizes that gold has appreciated during the last few years, or what is the same, that prices of other goods have fallen.[2]

In addition to his work on the theory of value, Cournot developed theories of exchange, monopoly, and competition.

These mathematical researches have been carried on and others have been started by such men as W. S. Jevons, Léon Walras, Vilfredo Pareto, F. Y. Edgeworth, A. L. Bowley, Luigi Amoroso, Alfonse de Pietro-Tonelli, Irving Fisher, H. L. Moore, Ragnar Frisch, G. C. Evans, Henry Schultz, and Harold Hotelling.[3]

The literature of mathematical economics is now quite extensive. In fact, it is too extensive to be surveyed adequately in a short paper. Professor Irving Fisher made a general survey of the contributions of mathematics to economics in the Gibbs Lecture for 1929 which he delivered before the American Mathematical Society.[4]

In the present paper one particular aspect of mathematical economics is considered, the study of demand.

This study deals chiefly with the effects of past prices, present price, and expected future prices, upon demand. It is contended that the effects of these quantities can be best considered by using the theory of integral equations, since the very definition of an integral makes it a function of all points of a curve—for the demand problem the curve represented by price as a function of time.

In section 2, it is postulated that demand depends upon present and past prices, and demonstrated that this assumption leads quite naturally to an integral equation of demand as indicated by formula (1), or to a functional equation of demand such as (2) or (3). In the simplest case, that is, for a Volterra integral equation of demand of type (1), the statements (i) demand depends upon past prices, and (ii) price depends upon past demands, are shown to be entirely equivalent, because of the invertibility of the integral equation.

In section 3, present demand is assumed to consist of (a) consumma-

[2] See, for example, Irving Fisher, *First Principles of Booms and Depressions*, New York, 1932.

[3] For bibliographies see (1) Irving Fisher's appendix to Cournot, *Researches into the Mathematical Principles of the Theory of Wealth;* (2) G. C. Evans, *Mathematical Introduction to Economics*, New York, 1930; (3) O. Weinberger, *Mathematische Volkswirtschaftslehre*, Leipzig and Berlin: B. G. Teubner, 1930; (4) Alfonso de Pietro-Tonelli, *Traité d'économie rationelle* (translated from the Italian by R. Gambier) Paris, 1927. For a comprehensive development of contributions made during the period 1844–1924 see A. L. Bowley, *The Mathematical Groundwork of Economics*, London, 1924.

[4] Irving Fisher, "Applications of Mathematics to the Social Sciences," *Bull. of the Amer. Math. Soc.*, April 1930.

tions of past decisions to purchase as soon as funds become available, (b) climaxes of growing desires to purchase, and (c) purchases that are called forth chiefly by the present price of the goods. For (a) many will make their purchases within a relatively short time after deciding to purchase, others will require more time, and some will never be able to purchase. This situation may be typified by assuming that the factors affecting the periods of time required for the transformation of desires to purchase created at time t_i, into actual purchases at t, are such that decisions to purchase are transformed into purchases according to a Pearsonian Type III frequency function. For this function and certain other somewhat similar ones, it is shown that the continuous weight function for past prices is of the type $f(x)K(x-t)$ where x refers to past time and t to present time. The form of this function is derived. For (b) it is shown that the demand depends less and less upon past prices as the time is more and more remote and a typical form of weight function is suggested.

In section 4 it is assumed that speculative demand depends upon expected future prices and that the demand law for the common situation of predicting prices one unit in advance of t consists of a function of p, a functional of p, and an expression involving the derivative of p with respect to time t. In section 5 a discussion of time lags is given and in section 6 a functional elasticity of demand is defined. This is a new concept.

Much of the analysis is presented here for the first time. Acknowledgments are made to other authors whenever their contributions are known. It is believed that the references are fairly complete, but probably not exhaustive.

II. FACTORS INFLUENCING DEMAND

A practical theory of demand will take all influencing factors into account, but it will *emphasize* only the important factors. Price is an important factor, but it is not enough to say that demand depends upon price. There are other very important variables, for example, the influence of past prices, speculation, prices of competing or substitute goods, monetary and credit conditions, taxes, and risk (which may include disaster or obsolescence). When an attempt is made to take into account all these factors, or even a few of them, the tremendous complexity of economic theory becomes apparent. An example is instructive.

Let $p(t)$ represent the average retail price of a commodity U at the time t, and suppose that the commodity has been offered continuously in a *particular* market for some time. Let $y(t)$ represent the amount purchased for consumption or speculation at the time t. In a general

theory of demand the time for consumption of the goods would undoubtedly be an important factor that should be taken into account. It is easier and less confusing, however, to build a theory without this factor, and then to extend the theory.

Consider, first of all, goods for which there is no buying for speculative purposes. This is practically equivalent to specifying that the problem be limited to retail buying and selling, that is, to consumers' demand, since there is considerable speculative activity associated with nearly all, if not all, commodities in the wholesale markets. After this theory has been developed it is easier to include the speculative factor.

Non-speculative demand for goods will certainly depend upon the price of the goods as well as upon the prices of competing or substitute goods, that is, it will be a function of the present price $p(t)$, the past prices $p(t_i)$, $i=0, 1, \cdots, n-1$, $t_0=0$, $t_n=t$, and the prices of m competing goods $p_1, \cdots, p_m$. Theoretically, it would also be a function of the past prices of competing or substitute goods. There is, however, no objection to simplifying the problem at first by assuming that the past price of each competing or substitute goods, $pk(t_i)$, $k=1, \cdots m$, is equal to p_k. The demand will also, in general, vary with the seasons and changes in social taste, which may be assumed to include obsolescence, etc., that is, demand will depend explicitly upon time t. The above may be summarized by taking the demand as a function

$$y(t) = f(p(t_0), p(t_1), \cdots, p(t_{n-1}), p(t), p_1, \cdots, p_m, t).$$

As a first approximation it might be assumed that $f(t)$ is a linear function of each $p(t_i)$, so that

$$y(t)=\phi(t)+[K(x_0, t)p(x_0)+\cdots+K(x_{n-1}, t)p(x_{n-1})]+K(x_n, t)p(x_n)$$

where, for simplicity in writing, the constant (with respect to time) or parametric arguments, $p_1, \cdots, p_m$, of ϕ, and K have not been written. Here, $K(x, t)$ is a weight function and the expression in brackets is the weighted average of the prices $p(t_0), \cdots, p(t_{n-1})$, in which, for the sake of clarity in the work that follows, x_i has been used in place of t_i.

There is no essential loss of generality by assuming $x_i-x_{i-1}=1$. If this assumption is made, it follows readily that the bracketed expression is the sum of rectangles of unit width and height $K(x_i, t)p(x_i)$. An application of the definition of a definite integral shows that this sum may be replaced by a definite integral, so that

$$(1) \qquad y(t) = \phi(t) + \int_0^t K(x, t)p(x)dx + K(t, t)p(t),$$

where $K(x, t)$ is a discontinuous step function. There is, however, no

objection to replacing $K(x, t)$ by a continuous function yielding the same integral, or for that matter, by one yielding approximately the same integral. In the work that follows $K(x, t)$ will be assumed continuous.

There are some examples for which $K(t, t) \equiv 0$. Thus, the demand for goods bought through a broker "at market" depends upon past prices, but not upon the price at the time t. However, in general, demand depends upon the present price as well as upon past prices and $K(t, t) \not\equiv 0$. In fact, $K(t, t)$ is not zero on the interval 0 to t. Mathematically, there is an important distinction between these two cases. For the first case, equation (1) becomes a Volterra integral equation of the *first kind*,[5] whereas for the second case it is a Volterra integral equation of the *second kind*.[6]

It is quite natural to ask if, conversely, price depends upon past demand. The answer is that whenever demand depends upon past prices, present price depends upon past demands, and the two statements are equivalent. In fact, those familiar with the theory of integral equations will recognize that equation (1) yields this dual theory. As is well known, if $K(t, t) \neq 0$ in the interval t_0 to t, equation (1) can be written in the completely equivalent form

$$K(t, t)p(t) = y(t) + \phi(t) + \int_0^t k(x, t)\phi(x)dx + \int_0^t k(x, t)y(x)dx,$$

where $k(x, t)$ is $K(tt)/K(xx)$ times the resolvent kernel of $K(x, t)/K(t,t)$ and as such is completely determined as a function of x and t.[7]

If the dependence of demand upon past prices is not linear, some other form of functional equation may be used. In some cases it may be possible to employ Volterra's functional analogue of Taylor's theorem[8] and write the demand as

$$y(t) = \phi(t) + K(t, t)p(t) + \int_0^t K_1(x_1, t)p(x_1)dx_1 + \frac{1}{2!}\int_0^t K_2(x_1, x_2, t)p(x_1)p(x_2)dx_1dx_2 + \cdots. \tag{2}$$

In other cases it may be possible to use a functional of the type

$$y(t) = \phi(t) + K(t, t)p(t) + \int_0^t K(x, t)\psi(p(x), x, t)dx \tag{3}$$

[5] Vito Volterra, *Leçons sur les équations intégrales* (Paris, 1913) p. 56.
[6] *Ibid*, p. 40.
[7] Vito Volterra. *loc. cit.*, p. 45.
[8] P. Levy, *Leçons d'analyse fonctionelle* (Paris, 1922), pp. 103–105.

where ψ is a non-linear function of p. Mathematicians have not yet been able to find out a great deal about non-linear integral equations. This, however, need not greatly concern the econometrician, for, in all liklihood, the present generation will not exhaust the cases for which (1) will be sufficiently general.

III. DEPENDENCE OF DEMAND UPON PAST PRICES

In the preceding analysis it has been assumed that demand depends upon past prices, but no explanation of the mechanism by which this relationship is brought about has been given. As a further problem it is now proposed to investigate the nature of the function $K(x, t)$. A study of why demand depends upon past prices, and how individuals build up wants or desires, throws considerable light upon the nature of the function $K(x, t)$.

Consider, first, goods of a semi-luxury nature, that is, goods which need not be purchased for the sake of existence or because of long established habit or custom. If desired, the influence of habit on demand can be considered later.

Suppose that, at the time $t_i<t$, a group consisting of G individuals, who have not decided to purchase a particular good, observed its price to be $p(t_i)$. Suppose further that this group purchased M_{t_i} units and decided to purchase a total of N additional units of the goods as soon as money or credit becomes available to them. Some of the group, perhaps, decided not to purchase any units while others probably decided to purchase more than one unit. Some, perhaps, purchased at the time t_i, and decided to purchase other units in the future. The tacit assumption is made here that the price does not increase more than some small amount (depending upon the particular group under consideration) before the consummation of the purchase.

For definiteness, suppose that the factors affecting the periods of time required for the transformation of desires to purchase into actual purchases, are such that decisions to purchase are translated into actual purchases according to a Pearsonian Type III frequency function.[9] For this type of function the amount $v(t, t_i)$ purchased by the group G at the time $t>t_i$ is given by the equation

$$v(t, t_i) = v_0 e^{-\gamma(t-t_i-\mu)}\left(\frac{t-t_i}{\mu}\right)^{\gamma\mu}, \tag{4}$$

where v_0, the mode of purchases, is proportional to N, that is, $v_0=\lambda N$, and where N is the total number of proposed purchases and γ and μ are parameters not involving the time t.

[9] See, for example, H. L. Rietz, *Mathematical Statistics*, 1927, p. 54.

The curve (4), (t_i, γ and μ constants) has a zero at t_i, reaches a maximum μ units of time after t_i and then approaches the t-axis asymptotically. Thus, it is assumed that many of the group will be able to make their purchases within a relatively short time (for some products one week, for others a month, and for some perhaps, a year) after t_i; others will require more and more time, and some will never be able to make their proposed purchases. The parameters γ and μ quite evidently depend upon monetary and credit conditions, upon the prices of competing or substitute goods, and upon various other quantities as well. For example, convincing advertising of the product presented to the group at the time t_i may cause certain of the group to do without other products sooner than they would if the advertising were not so alluring.

As pointed out above, v_0 is proportional to N, that is, $v_0 = \lambda N$, where $\lambda = (\gamma\mu)^{\lambda\mu+1}\mu\, e^{\lambda\mu}\Gamma(\gamma\mu+1)$ where $\Gamma(\gamma\mu+1)$ is the gamma function for $(\gamma\mu+1)$.[10] The number of proposed purchases N, will certainly be a function of the price $p(t_i)$. It would seem proper to say that individuals decide to make future purchases in accordance with laws of demand (functions of price) similar to the ones by which they make present purchases not influenced by past prices. Thus, in order to give N a practical form, one should examine the work done in finding statistical laws of demand as functions of price. Such an examination of the statistical literature of demand indicates that there are several likely forms, for example,

1.	$N = ap + b$	$a < 0, b > 0$
2.	$N = b/p^a + c$	$b > 0; c \geqq 0,$
3.	$N = b - \sqrt{ap - c}$	$b > 0, c > 0, a > 0,$
4.	$N = ap^2 - bp + c$	$a > 0, b > 0, c > 0,$
5.	$N = p^a e^{b(p-c)}$	$a > 0, b < 0, c > 0, e = 2.718 \cdots.$

Formula 1 represents a straight line with negative slope. H. L. Moore[11] used this form to obtain a statistical law of demand for cotton. Henry Schultz[12] employed it to obtain statistical laws of demand for sugar in terms of link relatives and trend ratios. G. P. Scoville[13] used it in studies of potatoes and hay. For small price variations this form must be considered to be a fair approximation for almost any type of demand law.

[10] See, for example, D. C. Jones, *A First Course in Statistics*, p. 221.

[11] H. L. Moore, *Forecasting the Yield and Price of Cotton*, 1917, p. 143.

[12] Henry Schultz, *Statistical Laws of Demand and Supply*, 1928, p. 61.

[13] G. P. Scoville, "Per Cent of the Expected Crop Correlated with Purchasing Power of the Price of Potatoes for Each of the Past Fifty-four years," Cornell University, Dept. Farm Marketing, 1919.

Formula 2 defines a hyperbola which approaches the demand axis and the price $p=c$, asymptotically. For $c=0$ this becomes the law, $y=b/p^a$, which was popularized by Alfred Marshall[14] and used in statistical work by W. P. Heddon,[15] J. Bloxom,[16] and H. H. Holland.[17] It has undoubtedly led to the common statements that the amount demanded is infinite when the price is zero and the price is infinite when the demand is zero. These statements must surely have been brought about by attempts to popularize a mathematical formula or by misconceptions of demand. In this world there are no goods of which an infinite amount is available, and if there were an infinite amount, there is neither an infinite number of consumers nor a finite number who could consume an infinite amount. Furthermore, if there were a very large supply, which, for practical purposes, might be considered to be infinite, as, for example, the supply of salt, the price would not be zero since handling or various other distributing charges would be involved. In other words, if all goods were supplied like manna there would still be the task of distributing the goods. G. F. Warren and F. A. Pearson[18] have taken this latter observation into account by assuming that $c\neq 0$. Of course, in spite of the above discussion it must be admitted that for values of p and y not close to zero, a law of the type 2, with c zero or not zero, may give a better approximation of a law of demand than equation 1. Y here is identical with N.

Equation 3 represents the lower branch of a parabola open to the right, whose vertex is at $y=b$, $p=c/a$. Thus, y is not defined for $p<c/a$. This curve therefore typifies the situation for which there is a finite demand $y=b$ at a minimum price $p=c/a$, which price may be taken to be the price of distribution. Furthermore, the demand is zero when the price $p=(c-b^2)/a$. Theoretically, the demand should remain zero for $p>(c-b^2)/a$, so that formula 3 has no significance for p greater than this value.

Formula 4 represents a parabola open above with vertex at $p=+b/2a$, $y=c-b^2/2a$. This curve crosses the y-axis at $y=c$ and may or may not cross the p-axis depending upon the values of a, b, and c.

Formula 5 defines a curve passing through the origin and approach-

[14] Alfred Marshall, *Principles of Economics*, 1920. For a theoretical derivation of the law $y=b/p^a$ see V. Pareto, Mathématique Économie, *Encyclopédie des Sciences Mathématique*, Tome I, p. 617.

[15] W. P. Heddon, "Measuring the Melon Market," *U. S. Dept. of Agr. and Port of New York Authority*, August 1924.

[16] J. Bloxom, "Some Determining Factors in Apple Prices," New York Food Marketing Research Council, *Food Marketing Studies*, I, 1926.

[17] H. H. Holland, "The Demand for Peaches in New York," *ibid*, III, 1926.

[18] G. F. Warren and F. A. Pearson, *The Interrelationships of Supply and Price*, Cornell University, 1927.

ing the p-axis asymptotically on the right. Thus, the curve rises from the origin, reaches a maximum (but the slope is not zero at the maximum), and then drops again to zero. H. L. Moore,[19] Henry Schultz,[20] and H. B. Killough,[21] have used this curve in statistical studies.

In the above discussion, the quantities a, b and c have been regarded as constants. For greater generality one might suppose them to be functions of time t, and other economic factors. The important point to remember is that for typical economic situations, dN/dp will be negative.

As a first approximation suppose that $N=a(t)p(t)+b(t)$. Then, the total amount, $y(t)$, purchased by all groups at the time t will be given by $y(t)=\sum_{i=0}^{n-1} v(t, t_i)+M_t$, or, that is,

$$y(t) = \sum_{i=0}^{n-1} \lambda\left[a(t_i)p(t_i) + b(t_i)\right]e^{-\gamma(t-t_i-\mu)}\left[(t - t_i)/\mu\right]^{\gamma\mu} + M_t.$$

Now, the sum may be closely approximated by an integral, so that the total amount purchased by all groups at the time t may be taken as

$$y(t) = \int_0^t \lambda\left[a(x)p(x) + b(x)\right]e^{-\gamma(t-x-\mu)}\left[(t - x)/\mu\right]^{\gamma\mu}dx + M_t.$$

Now, M_t consists of units purchased at t that were not predestined to be purchased at that time and decreases due to price increases at t. For example, an individual in the group $G(t_i)$ may have decided that he would purchase four units at \$1.00 each, but, at \$.85 each, he would change his decision and purchase five units, the latter number being better suited to his needs. Again, individuals who had not decided to make any purchases might purchase at t for a price $p(t)$. If it be considered that the desires of this latter group had been growing ever since the time t_k, the amount purchased by the group would depend upon past prices as well as upon present price, depending less and less on past prices as the time is more and more remote. Thus, one might write[22]

$$M_t = M(p(t), t) + \int_0^t h(x)e^{-w_1(t-x)}p(x)dx$$

[19] H. L. Moore, "The Elasticity of Demand and Flexibility of Prices," *Jour. Amer. Stat. Ass'n*, Vol. 23, 1922, pp. 8–19.

[20] Henry Schultz, "The Statistical Measurement of the Elasticity of Beef,' *Jour. Farm Econ.*, July 1924.

[21] H. B. Killough, "The Price of Oats," *U. S. Dept. Agr. Dept. Bull. 1351*, pp. 8 seq., Sept. 1925.

[22] See, G. C. Evans, *Mathematical Introduction to Economics* (New York, 1930), p. 46.

where M is a function of p and t, $h(t)>0$ and $w_1>0$. The function M might be assumed to obey one of the laws (5), for example, 1), so that,

$$(6)\ y(t) = \phi(t) + \int_0^t K(x, t)p(x)dx + \int_0^t \overline{K}(x, t)p(x)dx + a_1(t)p(t),$$

where

$$\phi(t) = \int_0^t \lambda b(x)e^{-\gamma(t-x-\mu)}\left(\frac{t-x}{\mu}\right)^{\gamma\mu}dx + b_1(t),$$

$$K(x, t) = \lambda a(x)e^{-\gamma(t-x-\mu)}\left(\frac{t-x}{\mu}\right)^{\gamma\mu}$$

$$\overline{K}(x, t) = h(x)e^{-w_1(t-x)}$$

$$M(p, t) = a_1(t)p(t) + b_1(t) \qquad a_1 < 0,\ b_1 > 0.$$

Formula (6) is simply a special case of formula (1) deduced in a slightly different way. The parameters in $K(x, t)$ and $K_1(x, t)$ quite evidently depend upon budgetary considerations. Thus, a great deal of the demand represented by the term $\int_0^t K(x, t)p(x)dx$, or by $\int_0^t K_1(x, t)p(x)dx$, may be contributed by custom or habit. For example, the function $b(x)$ may be a periodic function of time x. Similarly, $h(x)$ may be a periodic function, but it is not necessary that b and h be periodic functions of time. However, there is insufficient space here to develop this idea at length.[23]

Quite obviously different functions $K(x, t)$ can be obtained by assuming different distributions for the consummation of purchases at the time t. Thus, the distribution

$$v = v_0\left[1 + \frac{(t-t_i-\mu_1)}{\mu_1}\right]^{m_1}\left[1 - \frac{(t-t_i-\mu_2)}{\mu_2}\right]^{m_2\lambda, m_1/\mu_1 = m_2/\mu_2,}$$

and the same linear law of demand decision would lead to

$$K(x, t) = \lambda_1 a(x)[1 + (t-x-\mu_1)/\mu_1]^{m_1}[1 - (t-x-\mu_2)/\mu_2]^{m_2},$$

[23] In a book *Dynamical Economics*, now in preparation, the author has developed the ideas of this paper in much fuller detail. Here an adequate discussion of the formation of demand laws from budgetary data is given. For the present it is hoped that the mathematical reader will be able to take the suggestions given in this paper and develop them to his own satisfaction. For theoretical work on budgetary demand functions see A. C. Pigou, *Economics of Welfare*, Appendix II, and James W. Angel, "Consumers' Demand," *Quarterly Journal of Economics*, (1925), pp. 584–611. See also A. C. Pigou, "The Statistical Derivation of Demand Curves," *Economic Journal*, XL, 384–400, and Erich Schneider, "Demand Curves of Monopolized Commodities Derived from their Marginal Cost," *Archiv fur Sozialwissenschaft*, December 1931.

where

$$\lambda_1 = \frac{m_1^{m_1} m_2^{m_2} \Gamma(m_1 + m_2 + 2)}{(m_1 + m_2)^{m_1+m_2} \Gamma(m_1 + 1)\Gamma(m_2 + 1)} \frac{1}{\mu_1 + \mu_2}$$

From the point of view of shape and adaptability to curve-fitting, a particularly interesting type of curve to represent demand decision is

$$v(t, t_i) = (e^{\alpha(t-t_i)} - e^{\beta(t-t_i)})\gamma(t_i)$$

where α and β are negative constants and γ a positive function of t_i. For α, β, t_i fixed v represents a curve with a zero at $t = t_i$, a maximum at $t = t_i - (\beta/\alpha)^{1/\alpha-\beta}$ and an asymptote at $v = 0$. This $v(t, t_i)$ leads to a kernel of the simple type

$$K(x, t) = \gamma(x)(e^{\alpha(t-x)} - e^{\beta(t-x)})$$

Let t_0 be the time over which the effects of past prices are appreciable. Then $y(t)$ may be written in the form

$$y(t) = ap(t) + b + \lambda \int_{t-t_0}^{t} \gamma(x)\left[e^{\alpha(t-x-t_0)} - e^{\beta(t-x-t_0)}\right]p(x)dx.$$

By taking α, β, or γ proportional to national income or income affecting the industry under consideration, variable lags depending on monetary and credit conditions can be introduced. This gives an important tie-up of general economic theory with monetary theory. The author has developed this idea at considerable length in a study of commodity dollars, the effects of price fixing, and so forth.

Different kernels, $K(x, t)$, can also be obtained by assuming laws of demand decision other than the linear one which has been assumed here. There are, thus, a variety of forms, $K(x, t)$, that are theoretically possible.

IV. PRICE FORECASTING AND SPECULATION

An analysis similar to the preceding can be used to determine the form a demand law should take when speculation is taken into account. Speculative demand depends upon future prices, so that demand for goods that lend themselves to speculation depends upon past prices, present price, and expected future prices, that is, the amount demanded at the time t is

$$y(t) = f(p(t_1), \cdots, p(t_n)p(t_{n+1}), \cdots, p(t_{n+\sigma}), tp_1, \cdots, p_m),$$

where t_n stands for time t when purchases are made and $p(t_{n+1}), \cdots, p(t_{n+\sigma})$ are expected future prices. The analysis already developed shows that, for hypotheses similar to those made for obtaining equation (1), this expression can be replaced by an integral,

$$y(t) = \phi(t) + \int_0^T K(x, t)p(x)dx$$

where $T = t_{n+\sigma}$. More generally, since special weight may be attached to the present price $p(t)$,

$$y(t) = \phi(t) + \int_0^T K(x, t)p(x)dx + a_1(t)p(t).$$

This expression can be written as

$$(7)\quad y(t) = \phi(t) + a_1(t)p(t) + \int_0^T K(x, t)p(x)dx + \int_0^T \psi(x, t)p(x)dx,$$

where, for purposes of clarity, $K(x, t)$ for $x > t$ has been named $\Psi(x, t)$. The first three terms of (7) can be taken to be the same as the expression (1) or (6). Thus, $S = \int_t^T \Psi(x, t)p(x)dx$ represents demand due to speculation.

Long range economic forecasting has never been consistently successful. In fact, in view of the recent study by Cowles[24] of the measure of successes of economic forecasters, it is doubtful that any measure of success, other than what might be expected by chance, can, at present, be obtained for even short-time forecasting. Nevertheless, many individuals seem to be engaged in the science (or guessing game) of attempting to predict what will happen to prices the next minute, the next hour, the next day, or the next week. An important speculative problem is, therefore, the one for which $T = t+1$. For this problem, the extrapolation curve for $p(x)$ may be most conveniently taken as a straight line $p(x) = p(t) + (x-t)dp/dt$, $x > t$, where dp/dt is the derivative of price with respect to time, that is, dp/dt is the slope of the price-time curve at the time t. A substitution of this value of $p(x)$ in S yields,

$$S = \int_t^T \psi(x, t)p(x)dx = \int_t^{t+1} \psi(x, t)\left[p(t) + (x - t)dp/dt\right]dx.$$

An application of the first law of the mean for integrals[25] followed by an integration gives

$$S = \psi(t + \theta, t) \int_t^{t+1} \left[p(t) + (x - t)dp/dt\right]dx$$

$$= \psi(t + \theta, t)\left[p(t) + \tfrac{1}{2}dp/dt\right], \qquad 0 < \theta < 1.$$

[24] "Can Stock Market Forecasters Forecast?" by Alfred Cowles 3rd, ECONOMETRICA, I, No. 3, (July 1933), 309–324.

[25] E. W. Hobson, *Theory of Functions of a Real Variable* (Cambridge, 1927) I, 617. Continuity of $\psi(x, t)$ in x for all t is assumed.

It follows that equation (7) can be replaced by the simpler form

$$(8) \qquad y(t) = \psi(t) + A(t)p(t) + H(t)dp/dt + \int_0^t K(x,t)p(x)dx,$$

where $A(t) = a_1(t) + \Psi(t+\theta, t)$ and $H(t) = \Psi(t+\theta, t)/2$.

In general, when prices are rising and there is reason to believe that they will continue to rise, speculative demand is positive. When prices are falling and there is reason to believe that they will continue to fall, speculative demand is negative, that is, there is a tendency to dump. To satisfy either or both of the above conditions, $H(t)$ must be a positive function.

Many times prices rise for a period, drop back slightly, rise a bit and again drop back to the preceding recession level. In such instances there is uncertainty regarding trends, that is, there is uncertainty whether prices will rise of fall over a period of time. Some speculative traders sell to avoid losses, or sell "short" expecting to buy back at lower prices; some traders buy in the expectation of gain, and some simply stay out of the market. This situation is characterized by the fact that $dp/dt = 0$, that is, demand is neither increased nor decreased by speculation.

The introduction of a derivative of price into the demand equation marks one of the important contributions of mathematics to economics in the past decade. This is due to G. C. Evans, who in 1925 proposed a demand equation $y(t) = a\ p(t) + b + h\ dp/dt$, where a, b and h are constants, in order to approximate the phenomenon that "when prices are going up the demand (for lumber) is insatiable, but when prices are going down it is nil until the price movement stops."[26] Recently Professor Henry Schultz and his student R. H. Whitman, in studying the demand for pig iron[27] have successfully used a demand law of the type (8) with constants replacing the various functions ϕ, A, K and H.

It will be recalled that in section 2 the prices $p_1, \cdots, p_m$ of m goods, competing with the goods whose price is p, were assumed to be constant with respect to time. Since the theory just developed is a dynamical one, that is, one in which forces are allowed to modify situations as time progresses, and, hence, one in which the time element plays an important rôle, it was unnecessary to carry these parameters along in the equations. It should be remembered now, however, that in accordance with the hypotheses made in section 2, each of the quantities

[26] G. C. Evans, "The Dynamics of Monopoly," *Amer. Math. Monthly*, XXXI (1924), p. 77. See also Evans, *Mathematical Introduction to Economics*, New York, 1930. For the use of an integral equation of demand, see C. F. Roos, "A Mathematical Theory of Competition," *Amer. Jour. of Math.*, XLVII (1925), 173.

[27] ECONOMETRICA, I (1933), 94–96.

ϕ, A, H, and K, depends upon the prices of competing goods. For example, suppose that in a study of the relation of the demand for pig iron to the price of pig iron, the coefficient $A(t)$ is determined statistically as a number a. This number a, on further analysis, will be found to be made up of prices of competing goods such as copper, aluminum, lumber and so forth. Thus a might be defined by a formula such as $a=ap_a+cp_c+lp_l$, where a, c and l are constants and p_a, p_c and p_l are the prices of aluminum, copper and lumber respectively.[28] More generally a, c and l may be assumed to vary with the time t so that a will be a function of time. Thus, as far as the whole of the preceding analysis is concerned, the quantities ϕ, A, H, and K, may be considered to be functions of present prices of competing or substitute goods and the time t either implicitly or explicitly. For seasonal goods ϕ and possibly other quantities will contain periodic functions of the time, i.e., cosines and sines.

The statements made above in regard to changes in ϕ, A, H, and K, with respect to time, do not mean, however, that the quantities are so continuously changing that statistical laws of demand can not be determined. On the contrary, due to compensation of prices of competing goods, these quantities may remain fairly constant for periods of time that may be in excess of three to five years for some commodities. In the example considered above, it might be possible for the price of aluminum to increase, the prices of copper and lumber to decrease, and yet a might remain constantly equal to some given value, but such a situation would probably not prevail for long. In other words, econometricians might do well to devote their studies to daily or weekly data instead of to yearly data.

If the prices $p_j(t_i)$, $j=1, 2, \cdots, m$, $i=0, \cdots, n$, $t_0=0$, $t_n=t$, are not assumed to be all equal to $p_j(t)$ respectively, the theory of demand becomes much more complicated, but, nevertheless, it can be readily formulated in terms of systems of integral equations or functional equations. Thus the simplest case of the non-speculative problem leads to a system of $m+1$ integral equations,

$$y_k = \sum_{j=1}^{m+1} \left[\psi_{kj}(t)p_j(t) + \int_0^t K_{kj}(x, t)p_j(x)dx \right], \quad k = 1, \cdots, m+1.$$

There is, in general, no difficulty in inverting this system of Volterra integral equations to determine the $p_j(t)$ in terms of the y_k.[29] In fact, entirely similar equations result with the y_k replacing the p_k, and re-

[28] For a statistical study of the effects of prices of competing goods, see Mordecai Ezekiel, "Factors Affecting the Price of Hogs," *U. S. Agr. Dept., Bull.* 1440:37, 1926.

[29] Vito Volterra, *loc. cit.*, pp. 71–74.

solvent kernels replacing the K_{ij}. When derivatives are introduced and more general functional relations are assumed, a system of $m+1$ functional-differential equations is obtained. Such a general theory has already been developed.[30]

V. TIME LAGS

Suppose now that the first law of the mean is applied to the integral in (8), then

$$y(t) = \psi(t) + A(t)p(t) + H(t)dp/dt + G(t)p(\theta_1 t), \qquad 0 < \theta_1 < 1,$$

where $G(t) = \int_0 K(x, t)dx$. Finally, it is possible to combine the second and last terms and write

$$y(t) = \phi(t) + C(t)p((1 - \theta_2)t) + H(t)dp/dt, \qquad 0 < \theta_2 < 1.$$

The quantity θ_2 thus represents a time lag. From another point of view $p((1-\theta_2)t)$ may be considered to be an average price for the period of time 0 to t. It is important to notice, however, that it is not necessarily an arithmetic average. Furthermore, by using integral equations whose limits of integration extend from $t-t_0$ to t, variable lags are automatically introduced unless prices are constant.

For some agricultural products there is little speculative buying extending over a year, so that if the unit of time is a year, for these kinds of product $H(t)$ may be taken to be zero. Hence, the demand may be taken to be some function of a price previous to the present price or of an average price. Various more or less successful studies of demand for agricultural products have been made by a number of economists who have used yearly prices. Especially noteworthy studies have been made by H. L. Moore, who determined the first statistical law of demand, namely, that for cotton, by Henry Schultz, who determined a statistical law of demand for sugar, by G. F. Warren and F. A. Pearson, by Mordecai Ezekiel, by Holbrook Working, by E. J. Working, and by Louis Bean.[31]

VI. THE ELASTICITY OF DEMAND

For a great while economists have been classifying demand as elastic or inelastic according as the amount demanded increases much or little for a given fall in price and diminishes much or little for a given rise in price. In 1838 Augustin Cournot proposed to separate articles into

[30] C. F. Roos, "A Dynamical Theory of Economics," *Jour. Pol. Econ.*, XXXV, (1927), 648–650.

[31] For a bibliography through 1927, see G. F. Warren and F. A. Pearson, *The Interrelationships of Supply and Price*, Cornell University Publications, pp. 120–125. For an extensive up-to date bibliography, see *Price Analysis, Agricultural Economics Bibliography*, No. 48, U. S. Dept. Agr.

two categories according as $(\Delta y/\Delta p)p/y$ is less than or greater than one.[32] Alfred Marshall defined the quantity $(dy/dp)p/y$ to be the *coefficient of elasticity of demand*. He and H. L. Moore popularized the concept, and Moore applied it extensively to agricultural data and thereby obtained various laws of demand.[33]

Whenever the coefficient of elasticity of demand is *numerically* greater than one, the demand is said to be elastic. Whenever it is numerically less than one, the demand is inelastic and if it is numerically equal to one, the demand is neither elastic nor inelastic. For equation (8) the elasticity of demand is simply $A(t)p(t)/y(t)$.

For a functional demand equation of this type a functional coefficient of elasticity of demand might be more useful. Let δy be the variation of y corresponding to a variation, δp, in p in the sense of the calculus of variations. Then, the *functional coefficient of elasticity of demand* may be defined to be $(\delta y/\delta p)p/y$. Thus, for (8) with $H=0$, one would have $\delta y=\int_0^t K(x, t)\delta p dx+A(t)\delta p$. In particular, if each price of the goods from the time 0 to t were increased by a constant amount δp, the functional coefficient of elasticity of demand would be $\left[\int_0^t K(x, t)dx+A(t)\right] p(t)/y(t)$. If H were not zero, there would be added a term $Hd(\delta p)/dt$ to δy.

VII. CONCLUSION

In concluding this study it is important to notice that no attempt has been made to determine why a demand decision curve should take the form postulated, or why the number making a demand decision should depend upon the price, or why N or M_t should take any of the forms postulated. It would of course be possible to pursue these questions as far as one liked by analyzing human behavior, utility, and so forth.

The size of a market can be increased by advertising. This can be taken into account by postulating that the number introduced to a product at a certain time depends upon the advertising expenditure, and then proceeding to determine how demand varies with advertising.

Again, if the product is one for which there will be a repeat demand, the number of prospective purchases will certainly depend upon the life of the product. Thus, the effects of obsolescence, risk of damage, seasonal changes, and other factors, could be studied. Also, the parameters of the functions $K(x, t)$, $\phi(t)$, etc., depend upon such things as monetary conditions, and the psychology of the buying public. However, all these factors deserve consideration in their own right and can not be adequately treated in a short paper. It may be hoped that it

[32] Augustin Cournot, *loc. cit.* pp. 52–54.

[33] See, for example, Alfred Marshall, *Principles of Economics* (London, 1920), and H. L. Moore, *Synthetic Economics*, (New York, 1930).

will be possible to push the analysis of demand further and further back to fundamental conceptions of behavior, but it must not be forgotten that empirical formulae will have to be introduced at some stage.

It might be said that the theory of demand has progressed to a point where it is questionable that further theoretical work should be done until statistical studies have been made to verify or disprove the hypotheses and conclusions so far reached. This statistical investigation is, in itself, a tremendous task. A new type of mathematical statistician will be required to make the studies. Nevertheless, there are reasons to be optimistic regarding the possibilities of the discovery of statistical laws of demand.

London, England

ECONOMETRIC PARAMETERS IN A STATIONARY SOCIETY WITH MONETARY CIRCULATION[1]

By J. MARSCHAK

Translation from German by M. KROST

I. PARAMETERS RELATING TO CIRCULATION

§1. THE following remarks apply, where not otherwise indicated, to a closed, stationary economy. We shall call a *stationary* economy one in which the total value of the stock of goods remains constant, and the distribution of this stock of goods between the different stages of production, as is explained in Part II, remains unaltered, if regular seasonal fluctuations are eliminated, i.e., if one always compares averages over properly selected periods of observation. By *money* we understand both notes and demand deposits.

In what follows, by *real value* of a physical quantity of goods is meant the number of units of productive factors required to produce it and (under stationary conditions) exchangeable for it. Since under stationary conditions the exchange relations between the different factors (labor, capital) are constant, a more detailed definition of the unit of productive factor chosen as a measure is not necessary. We may think of one hour of labor, or of the service of $100 for a year, etc.

§2. In the equation of exchange

$$m \cdot v = t \cdot p \tag{1}$$

the symbols have in this paper the following significance: m is the nominal money stock, consisting of notes and demand deposits; t is the real value of the volume of transactions per unit of time; p is the price of a unit of productive factor. Then the equation

$$v = \frac{t \cdot p}{m} \tag{2}$$

is the definition of the *velocity of circulation of money*, v.

§3. In this connection we wish to consider only those transactions occurring in the regular circuit flow of the stationary economy which bring

[1] This paper follows the lines of a paper delivered at the meeting of the Econometric Society in Paris, October, 1932, and an article, "Volksvermögen und Kassenbedarf," published in the *Archiv für Sozialwissenschaft und Sozialpolitik*, LXVIII (January, 1933), but attempts to make the propositions still more precise. For most important stimuli, I am indebted to the writings of Holtrop, von Hayek, and Robertson (for Part II), Neisser (for Part III), and A. Marget (for Part IV). Cf. also the criticisms of Neisser, and my reply to him, both in the *Archiv für Sozialwissenschaft*, LXIX (July, 1933).

commodities from stage to stage of the productive process and finally to the consumer, trade, apart from speculative alterations in inventory, being considered as part of the productive process. Thus, we eliminate from consideration transactions in real estate, corporate stocks, titles to debt, and also speculative trading in commodities, i.e., we include neither the real value of these transactions in the quantity t nor the sums of money held for the purpose in the quantity m. On the other hand, sales of productive factors are to be contained in t.[2] These definitions have led us, in order to avoid misunderstanding, to use, not the capital letters of the Fisher equation, but the corresponding small letters.

§4. The real value of the money stock or the real money stock

$$\mu = \frac{m}{p}, \text{ so that } v = \frac{t}{\mu}. \tag{3}$$

We let r_0 indicate the real value of the national income, which, of course, in the stationary economy, is equal to the real value of consumption. Further, we define the degree of differentiation δ, as

$$\delta = \frac{t - r_0}{r_0}, \text{ so that } 1 + \delta = \frac{t}{r_0}. \tag{4}$$

In the case of complete vertical concentration, where producers sell only to consumers (i.e., income-receivers and at the same time sellers of productive factors), $t = 2r_0$ and $\delta = 1$. The quantity

$$f = \frac{v}{1 + \delta} = \frac{r_0}{\mu} \text{ [cf. (3), (4)]} \tag{5}$$

is often called the efficiency of money (in the case of complete vertical concentration efficiency equals velocity of circulation, that is, $f = v$); it measures per given period of time the average number of times a piece of money is used for payments of income (or purchases of consumers' goods.) I may use for it the expression *circuit frequency of money*, in order to be able to use an analagous expression for goods. That is to say, if we give to the real value of all goods in the hands of producers and consumers, including both quickly and slowly depreciating goods and those in the process of physical transformation, the name

[2] We do not exclude the sales of productive factors (payments of income) since the sum of money held for that purpose in entrepreneurs' balances cannot well be excluded logically from the total sum of money m required in the stationary circuit flow. On this account the definition (4) of δ becomes only a little more complicated. Here we follow the definition of Holtrop (as against that of my paper "Volksvermögen und Kassenbedarf").

of real commodity stock or real wealth K, and if we replace in equation (5) the real money stock μ by K, we get a system of quantities analagous to that just described. [The equations (6′) and (6″) we give only to show the relation of this to other terminologies.]

(6) $F = \frac{r_0}{K}$ = circuit frequency of goods,

(6′) $\frac{1}{F}$ = circuit time of goods (probably identical with the Böhm-Bawerkian "average period of production"),

(6″) $$V = F(1+\delta) = \frac{r_0(1+\delta)}{K} = \frac{t}{K},$$

velocity of circulation of goods, in the sense of A. Marget, who, however, includes in K only goods "ready for sale" (and thus not goods in the process of transformation and machines in operation, etc.), a limitation which seems to us logically hardly achievable.

In the explanation of monetary phenomena, we have to take into account primarily *fluctuations* of stocks of goods which have a period shorter than the time required to wear out so-called "durable" capital and consumers' goods; hence for statistical purposes we may consider, instead of the total stock of goods (K), only "inventories," i.e., stocks of the goods which are more quickly used up (K_1). By this procedure possible changes in the rate of amortization are, of course, neglected, but, on the other hand, the errors always connected with an estimate of total wealth are avoided, and these may be of great importance relative to the magnitude of the changes. $F_1 = \frac{r_0}{K_1}$ is the circuit frequency of inventories (approximately).

§5. We now attempt to give a statistical survey of the parameters referred to for Germany in 1928. To be sure, here we do not have a closed stationary economy, since the balance of capital imports amounted to about 4 billion M., the increase of real wealth to about 10 billion M., the increase of prices (for consumers' goods) to about 2 per cent in the course of the year. Even if, because of this last circumstance, the price of a commodity exceeds the price of the units of productive factor required to produce it by the business profit, we will consider as the unit of productive factor the amount of commodities which could be bought in the year 1928 for 1 M., believing that the following survey is not essentially affected by the source of error just referred to. The quantities $(\overline{K}-K)$, $(L-L_1)$ and w are further defined below.

SOME ECONOMETRIC PARAMETERS FOR GERMANY, 1928

M = Reichsmark; R = the quantity of goods 1 M. would buy in the year 1928

		Units	Sources
p	1	$\frac{M}{R}$	
m	12	Billion M.	Marschak-Lederer, *Arch. f. Soz.*, LXVII, 399; excluding "financial circulation proper."
μ	12	Billion R.	
r_0	75	Billion R. per year	*Das deutsche Volkseinkommen*, Stat. Reichsamt, 1932
t	300	Billion R. per year	Marschak-Lederer, *op. cit.*, p. 394
K	250	Billion R.	*Statistik der Einheitswerte 1928–30*, Stat. Reichsamt; in addition estimated publicly owned wealth and private wealth in consumers' goods.
K_1	30	Billion R.	*Vierteljahreshefte zur Konjunkturforschung* 6, H. 1A, p. 69; estimated agricultural and consumers' stocks are added.
δ	3	1	
v	25	once a year	
f	6.25	"	
F	0.3	"	
F_1	2.5	"	
$\overline{K}-K$	6–7	Billion R.	cf. §9
$L-L_1$	5–6	"	
w	100–300	once a year	cf. §10

II. PARAMETERS RELATING TO PRODUCTION AND THEIR CONNECTION WITH THE MONETARY PARAMETERS

§7. Let us arrange all existing goods (imagining slowly depreciating goods divided into separate amortization-segments following one another through time) according to their *distance from consumption* x, where x is measured in units of time. At a given moment, τ, a good has the distance from consumption x if one may assume that the consumers' good produced from this good is consumed in the moment $\tau+x$. Here the good is evaluated only at the sum of the cost-elements (intermediate products and factors of production) already *paid for*, so that, for example, a week's labor is included in the value of the stock of goods only

after the completed payment of wages. (This assumption corresponds, so far as the services of labor are concerned, to the accounting principle of practical life.)

Now, if at the moment τ the sum of the real values of all goods whose distances from consumption lie between x and $(x+\Delta x)$ is designated by $\left|K_\tau\right|_x^{x+\Delta x}$ or, for the sake of brevity, by $K_\tau(x)$ we give to the *real production value at distance from consumption* x the limiting value

$$r(x) = \operatorname*{Lim}_{\Delta x \to 0} \frac{K_\tau(x)}{\Delta x} \tag{7}$$

so that the real stock of goods

$$K_\tau = \int_0^\infty r(x)dx \tag{8}$$

The function $r=r(x)$ we may call the *function of manufacture*, the area K described by it the *wealth area*. The function $r=r(x)$ has under stationary conditions the following properties:

$$\begin{cases} 1) & r(0) = r_0 \\ 2) & \operatorname*{Lim}_{x \to \infty} r(x) = 0 \\ 3) & \dfrac{dr(x)}{dx} < 0. \end{cases} \tag{9}$$

The difference

$$(\Delta r)_x = r(x) - r(x + \Delta x) \tag{10}$$

is called in the English and American statistics "(annual) value added by manufacture," in the German statistics "(jährlicher) Veredelungswert" or "(jährliche) Wertschöpfung" of the production process taking place in the interval $x+\Delta x \longmapsto x$. From (10) and (9) it follows that

$$\sum_{x=0}^{x=\infty} (\Delta r)_x = r(0) - r(\infty) = r_0$$

i.e., the sum of all values added by manufacture constitutes the national real income, a proposition which is, indeed, self-evident.

§8. Production, like the purchase and sale of goods and productive factors, does not go on regularly throughout the year and to some degree does not go on continuously, both for natural and conventional reasons. Hence the stock of goods $K(x)$ included in the interval $x+\Delta x \longmapsto x$ and with it the limiting value $r(x)$ change as the moment of observation τ changes. The curve of manufacture $r=r(x)$ undergoes temporal deformations. But the stock of goods K of the (stationary)

economy *as a whole* must remain constant; and thus the measures of real value shift merely between different intervals along the distance from consumption:

$$K_{\tau_1}(x) \gtreqless K_{\tau_2}(x), \quad r_{\tau_1}(x) \gtreqless r_{\tau_2}(x)$$

but

$$\int_0^\infty r_{\tau_1}(x)dx = \int_0^\infty r_{\tau_2}(x)dx = \cdots = K.$$

The point of time θ_x, of *seasonal maximum stock* in the interval $x+\Delta x|\!\!-\!\!-\!\!|\, x$ is defined by

$$K_{\theta_x}(x) \geqq K_\tau(x)$$

where τ is any moment of time whatever.

In what follows we indicate $K_{\theta_x}(x)$ by $\overline{K}(x)$ and correspondingly define:

$$\bar{r}(x) = \operatorname*{Lim}_{\Delta x \to 0} \frac{\overline{K}(x)}{\Delta x}, \quad \overline{K} = \int_0^\infty \bar{r}(x)dx. \tag{11}$$

The figure described by the curve $\bar{r}=\bar{r}(x)$ whose area $\overline{K}=\int_0^\infty \bar{r}(x)dx$ we call the *area of maximum stocks;* it must be kept in mind that the maximum values $\overline{K}(x)$ and $\bar{r}(x)$ appear in different points of time θ_x for different distances from consumption x. The difference

$$\overline{K}(x) - K_\tau(x) = \mu_\tau(x) \tag{12}$$

is the real value of the sums of money which the owners of the goods existing in the interval $x+\Delta x|\!\!-\!\!-\!\!|x$ must have on hand in the moment τ to bring their stocks to the maximum point by the gradual expenditure of money in the period of time between τ and θ_x. If stocks of money were held only for this purpose, the real value of the total money stock existing in the economy would be

$$\mu = \operatorname*{Lim}_{\Delta x \to 0} \sum_{x=0}^{x=\infty} \mu_\tau(x) = \int_0^\infty \bar{r}(x)dx - \int_0^\infty r_\tau(x)dx = \overline{K} - K \tag{13}$$

and from (11) this holds for any τ whatever. For statistical purposes (§5) one may say, roughly: $\overline{K}-K=\overline{K}_1-K_1$, where $\overline{K}_1$ and K_1 are the quantities analogous to $\overline{K}$ and K for quickly depreciating stocks (inventories).

§9. Thus the quantity of money is measured by the difference between the area of maximal stocks and the area of wealth. We have here defined θ_x and $\overline{K}$ as if stocks of goods fluctuated only because of regularly recurring causes. Suppose the period of such fluctuations to be of

the length T (one year, one complete cyclical movement, etc.) and the observed values of the function

$$\rho_i(x) = \bar{r}_i(x) - r_i(x), \tag{14}$$

(where $i = 1, 2, \cdots, h$), for different values of x in the periods of time τ to $(\tau+T)$, $(\tau+T)$ to $(\tau+2T)$ $\cdots$ to have a standard deviation, which can be expressed as function of x, say $\sigma(x)$. Then the additional reserve held for such fluctuations ("liquidity reserve") is an increasing function:

$$L = L\left[\int_0^\infty \sigma(x)dx\right], \quad \frac{dL}{d\sigma} > 0. \tag{15}$$

Another such function, which decreases when σ increases, viz.

$$L_1 = L_1\left[\int_0^\infty \sigma(x)dx\right], \quad \frac{dL_1}{d\sigma} < 0 \tag{15a}$$

represents the diminution of the necessary stock of money, dependent on the standard deviation of the bundle of curves $\rho(x)$ made possible by the operations of *banks* (for the cases in which $\overline{K}(x)$ suddenly increases) and *insurance organizations* (for the cases in which $K(x)$ suddenly falls).[3]

The function L expresses the relation between the liquidity reserve and the standard deviation of ρ_i, which is calculated for every value of x separately and then summed over x. The function L_1 expresses the fact that the diminution of the necessary stock of money brought about by the operation of banks and insurance institutions is the greater, the greater the certainty with which these institutions can make their forecasts of the prospective fluctuations of the *curve as a whole* $\rho(x)$.

$$\begin{aligned} \mu &= \overline{K} - K + L - L_1; \text{ for the symbols} \\ k &= \overline{K} - K \\ l &= L - L_1, \text{ we have} \end{aligned} \tag{16}$$

$$\mu = k + l. \tag{16a}$$

If only annual seasonal fluctuations (T = one year) are considered as "regular," k in Germany (1928) is probably 6–7 billion R. and correspondingly l is 4–6 billion R. Since, roughly (cf. §4)

$$k = \overline{K}_1 - K_1,$$

[3] Further questions which the credit economy presents, especially changes of the possibilities of substitution between money (notes and demand deposits) and credit instruments (bills, bank credit) are left out of account in the present discussion.

k may be calculated approximately from data relating to business volumes in retail trade, transportation, agriculture, etc., which show strong seasonal movements. Therefore k lies approximately between 6 and 7 billion R. (as for individual intervals of distance from consumption, about the following *average* fluctuations in stocks occur: retail trade $1\frac{1}{2}$, wholesale trade in consumers' goods 1, industries producing consumers' goods $\frac{1}{2}$, agriculture 2, fertilizer and agricultural machinery $\frac{3}{4}$ billion R. On the other hand $\mu = 12$ billion R. (roughly) so that $L - L_1$ must be between 5 and 6 billion R.).

§10. The real stock of money μ cannot sink below a definite minimum. If A, the owner of an average real stock of money μ_A, sells on the average the real value r_A per unit of time, it holds that

$$\frac{r_A}{\mu_A} \leqq w, \tag{17}$$

where w is the *technical maximum of the velocity of money* in the "circuit-flow" markets (i.e. excluding "the financial transactions," cf. §3). $1/w$ is then the technical minimum of a duration of payment, averaged for the different means of payment. Putting as specific technical minima (always excluding the money-market, etc.) for local bank-payments, interlocal bank-payments, and cash payments (allowing for the nights and the change-reserves): 2–3 days, 3–4 days, and 0–$\frac{1}{2}$ days respectively, and estimating the proportion between bank- and cash-payments in the "circuit-flow" markets as 1:1, we should find for Germany (1928) an average $1/w = 1$ to $2\frac{1}{2}$ days (cf. *Archiv für Sozialwissenschaft* LXVII, pp. 394–5, and LXIX p. 500). Since

$$t = \sum r_j$$
$$\mu = \sum \mu_j;\ (j = A, B, C \cdots)$$

it follows from (17) and (2)

$$\mu \geqq \frac{t}{w} \tag{18}$$

or, from (4):

$$\mu \geqq \frac{1}{w} r_0(1 + \delta). \tag{19}$$

We shall later (§12) show the possibility of the lower limit for μ lying still higher.

III. THE DEGREE OF DIFFERENTIATION

§11. Let us think of the production going on in the interval of distance from consumption x_3|——|x_1 within the concern A as being

made up of two stages x_3|——|x_2 and x_2|——|x_1; $(x_1<x_2<x_3)$. The real value of the finished product is $r_A(x_1)$ and that of the half-finished product is $r_A(x_2)$. Then a change by which the two stages would be carried on in independent concerns A_1 and A_2 must increase the volume of trade t by $\Delta t=r_A(x_2)$ and the quantity δ by $\Delta\delta=r_A(x_2)/r_0$. At the same time, in the interval of distance from consumption in question, the standard deviation $\sigma(x)$ of the individual observed or expected values of

$$\rho_i(x) = \bar{r}_i(x) - \dot{r}_i(x) \qquad \left\{ \begin{array}{l} i = 1, 2 \cdots \\ x_1 < x_2 < x_3 \end{array} \right\} \text{ (cf. §14)}$$

increases, for A_2 can now forecast A_1's intentions of buying with less assurance and A_1 can inform himself less accurately of the conditions under which A_2 intends to sell than when the business was carried on under a single management. Consequently (for $\Delta t\to 0$)

$$\frac{d\sigma}{dt} > 0; \text{ and since } t = r_0(1 + \delta) \text{ and } r_0 = \text{const.:}$$

$$\frac{d\sigma}{d\delta} > 0; \text{ further, since from (15) and (15a):}$$

$$\frac{dl}{d\sigma} = \frac{d(L - L_1)}{d\sigma} > 0: \text{ then}$$

$$\frac{dl}{d\delta} = \frac{dl}{d\sigma}\cdot\frac{d\sigma}{d\delta} > 0. \tag{20}$$

For a given δ, σ is affected, however, by other factors independent of δ which produce the irregular fluctuations of the quantity ρ_i. If s is the standard deviation of these fluctuations which are independent of δ then

$$\left\{ \begin{array}{l} \sigma = \zeta(s, \delta) \text{ and} \\ l = l(s, \delta) \text{ where} \\ \dfrac{\partial l}{\partial s} > 0, \quad \dfrac{\partial l}{\partial \delta} > 0. \end{array} \right. \tag{21}$$

The relationship between l and δ is, however, not always unique. One and the same value of t (and hence of $\delta=t/r_0-1$), may be made up in very different ways according to the form of the function of manufacture $r(x)$ and according to the distances from consumption x_A, x_B, x_C, $\cdots$, at which the real transactions r_A, r_B, r_C, $\cdots$, take place:

$$t = r_A + r_B + r_C + \cdots = r_M + r_N + r_P + \cdots = \cdots$$

Therefore, different values of $\sigma(x)$ or l may correspond to one and the

same t or δ, even with $s=$ constant. Nevertheless, from (19) and the following considerations the limits may be given within which it is possible for l to vary.

Let us select, of all possible values of l which (for a given value of s) correspond to a definite value of δ, the greatest ($\bar{l}$) and the smallest ($\underline{l}$) so that

$$\bar{l} \geqq l \geqq \underline{l}$$

and let us represent $\bar{l}$ and $\underline{l}$ as functions of δ:

$$\bar{l}(\delta) \geqq l(\delta) \geqq \underline{l}(\delta). \tag{22}$$

From (20) it is demonstrable that:

$$\frac{\Delta\bar{l}}{\Delta\delta} > 0; \quad \frac{\Delta\underline{l}}{\Delta\delta} > 0. \tag{23}$$

In fact, let us select of all possible compositions of a definite value of $t=t_0$ those two extremes: $\bar{t}_0$ and $\underline{t}_0$ at which $l_0=\bar{l}_0$ and $l_0=\underline{l}_0$ respectively:

$$\bar{t}_0 = r_{\bar{A}} + r_{\bar{B}} + r_{\bar{C}} \cdots$$

and

$$\underline{t}_0 = r_{\underline{A}} + r_{\underline{B}} + r_{\underline{C}} \cdots .$$

Let us increase $\bar{l}_0$ by $r_{\bar{M}}$ (by "vertical" division of one concern into two) or decrease $\underline{t}_0$ by $r_{\underline{M}}$ (by "vertical" integration of two concerns) so that

$$\overline{\Delta\delta} = r_{\bar{M}} : r_0 > 0$$
$$\underline{\Delta\delta} = -\, r_{\underline{M}} : r_0 < 0.$$

From (21) it then follows:

$$\bar{l}_0 + \Delta\bar{l}_0 > \bar{l}_0, \qquad \underline{l}_0 + \Delta\underline{l}_0 < \underline{l}_0$$

$$l(\delta + \overline{\Delta\delta}) > \bar{l}(\delta), \quad l(\delta + \underline{\Delta\delta}) < \underline{l}(\delta)$$

but as, at the same time, from (22)

$$\bar{l}(\delta + \overline{\Delta\delta}) \geqq l(\delta + \overline{\Delta\delta}), \quad \underline{l}(\delta + \underline{\Delta\delta}) \leqq l(\delta + \underline{\Delta\delta})$$

it follows that:

$$\bar{l}(\delta + \overline{\Delta\delta}) \geqq \bar{l}(\delta), \quad \underline{l}(\delta + \underline{\Delta\delta}) \leqq \underline{l}(\delta)$$

so that from (20):

$$\frac{\Delta\bar{l}}{\Delta\delta} > 0, \qquad \frac{\Delta\underline{l}}{\Delta\delta} > 0. \tag{23}$$

Accordingly, if δ is the only independent variable, i.e., if $s = \text{const.}$, all l's lie within a field bounded by the two upward-sloping lines $l = \bar{l}(\delta)$ and $l = \underline{l}(\delta)$. The distance $\bar{l} - \underline{l}$ increases as δ increases, since the number of possible combinations $(t = r_A + r_B + \cdots = r_M + r_P + r_Q \cdots = \cdots)$ increases.

§12. The field within which l can lie is further limited:

(1) by the straight line $\delta = \mathring{\delta}$, where $\mathring{\delta}$ is the smallest degree of differentiation practically occurring; theoretically, $\delta = 1$ (cf. §4); in which case there can be only *one* composition of t so that $\bar{l} = \underline{l}$ and the straight line reduces to a single point.

(2) by the lower limit constituted by the restriction put on the maximal velocity of money by the mechanism of payment; we may now state this lower limit (cf. §10) as follows: the division of a concern A into A_1 and A_2 must increase the total real supply of money at least by the sum necessary for the transactions of A_2:

$$\frac{r_A(x_2)}{w} = \frac{\Delta t}{w}$$

so that:

$$\Delta\mu \geqq \frac{\Delta t}{w}, \text{ or}$$

$$\Delta\mu = \frac{\Delta t}{w} + \alpha\Delta t, \text{ where } \alpha \geqq 0.$$

By the division of concerns with smaller and smaller transactions we reach the limiting value:

$$d\mu = \frac{dt}{w} + \alpha dt; \text{ consequently}$$

$$\mu = \frac{t}{w} + \alpha t + C; \text{ if } \mathring{\delta} \text{ is the smallest degree of differentiation practically occurring } (1 \leqq \mathring{\delta} \leqq \delta) \text{ and } \mathring{t} \text{ the corresponding volume of trade } [\mathring{t} = r_0(1 + \mathring{\delta})], \text{ then where } \delta = \mathring{\delta}$$

$$C = \mu - \frac{t}{w} - \alpha t = l(\mathring{\delta}) + k - \frac{\mathring{t}}{w} - \alpha \mathring{t} \quad \text{(cf. 16a)}$$

$$\mu = \frac{t}{w} - \frac{r_0(1+\mathring{\delta})}{w} + l(\mathring{\delta}) + k + \alpha(t - \mathring{t})$$

and, since $\alpha \geqq 0$ and $t - \mathring{t} \geqq 0$,

$$\mu \geqq \frac{t}{w} + l(\mathring{\delta}) + k - \frac{r_0(1+\mathring{\delta})}{w} \text{ ; or, more briefly,} \tag{24}$$

$$\mu \geqq \frac{t}{w} + \beta \tag{24a}$$

This lower limit is higher than that stated in §10

$$\mu \geqq \frac{t}{w} \tag{18}$$

by the positive quantity $\beta = \mathring{t}(1/v - 1/w)$, the lower limit of which itself depends on the function $l(\delta)$. In the theoretical case $\mathring{\delta} = 1$

$$\mu \geqq \frac{t}{w} + l(1) + k - \frac{2r_0}{w}.$$

We have represented this case in Figure 1. The field of possible l's (distances from the O-axis) or of possible μ's (distances from the O'-axis) is shaded in the figure. Further it is true (21), that with successive equal changes in t (or δ), l (or μ) must move along an upward sloping curve (if $s = \text{const.}$). Thus, if we examine, not two states which are not connected with each other, but a *process* of differentiation (or concentration) which consists only of a series of business divisions (or combinations), but not of a mixture of divisions and combinations, we can make use, instead of the widening field $l\,\overline{[\delta]}$, of the upward-sloping curve $ABCDE$ (see figure), which depends on the "path" taken by the successive processes of differentiation. This curve is continuous, so far as r_A can be very small in relation to t. However, a break can occur at the point C, where the line of technical minimum cash-holdings cuts the line of cash-as-affected-by-fluctuations-of-stocks.[4]

[4] I am much indebted to Dr. Karl Schlesinger, Vienna, who drew my attention to the fact that k (and not only l) is likely to depend on the degree of differentiation δ. We may, in fact, write (cf. (21)) $\frac{\partial k}{\partial \delta} > 0$, since we consider, for each interval between two distances of consumption, the regular fluctuations of the *full-paid* stocks only, and since such fluctuations evidently disappear for the

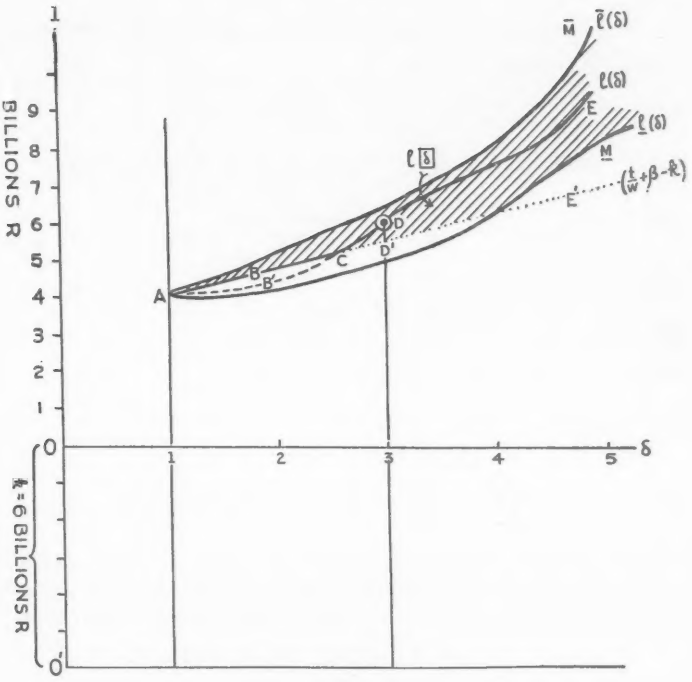

FIGURE 1

$OO' = k$ = real stock of money required because of regular fluctuations in the supply of goods.

$OL = l$ = real stock of money required because of irregular fluctuations in the supply of goods.

$OO' + OL = \mu$ = total real stock of money.

$A\overline{M} = \bar{l}(\delta)$ = line of maximal real stock of money required because of fluctuations in the stock of goods: $\mu = \bar{l}(\delta) + k$.

$A\underline{M} = \underline{l}(\delta)$ = line of minimal real stock of money required because of fluctuations in the stocks of goods: $\mu = \underline{l}(\delta) + k$.

$ABCD'E'$ = (straight) line of minimal real stock of money required by the mechanism of payment:

$$\mu = \frac{t}{w} + l(\mathring{\delta}) - \frac{r_0(l + \mathring{\delta})}{w} + k$$

$l\,\overline{|\delta|}$ = field of possible real supply of money (shaded).

$AB'C\bar{D}E$ = line of real money supply required because of fluctuations in the stocks of goods, through a given path of differentiation.

$ABCDE = l(\delta)$ = actual line of real money stock through a given path of differentiation: $\mu = l(\delta)$

C = point of intersection of $ABCD'E'$ and $AB'CDE$, where a break occurs in the line $l(\delta)$.

D = position in Germany, 1928 (roughly).

The values used are: $r_0 = 75$ billion R. per year; $\delta = \mathring{\delta} = 1$; $w = 150$ per year; the curves $\bar{l}(\delta)$, $\underline{l}(\delta)$, $AB'CDE$ are arbitrary.

IV. THE VELOCITY OF CIRCULATION OF MONEY AND THE PRICES

§13. In consequence of (5)

$$\mu = r_0 \cdot \frac{1+\delta}{v}. \tag{24b}$$

Taking logarithms and differentiating we have, where $r_0 = \text{const.}$

$$\frac{d\mu}{\mu} = \frac{d\delta}{1+\delta} - \frac{dv}{v} \tag{25}$$

$$\frac{d\mu}{dv} = \frac{\mu}{1+\delta}\cdot\frac{d\delta}{dv} - \frac{\mu}{v}; \tag{25a}$$

if $\delta = \text{const.}$

$$\frac{d\mu}{dv} = -\frac{\mu}{v} \tag{26}$$

and, if besides, $m = \text{const.}$

$$\frac{dp}{dv} = \frac{d\left(\frac{m}{\mu}\right)}{dv} = \frac{p}{v} = \frac{m}{t} = \text{const.} \tag{27}$$

which is also immediately evident from (1). The real problem, however, is offered by the general case where δ is a variable, when changes of v may be brought about by changes in δ *but also* by other causes.

$$\mu = k + l(s, \delta) \quad \text{(cf. (21), (16a))}$$

$$v = \frac{r_0(1+\delta)}{k + l(s, \delta)}$$

In the case of simultaneous variation of the independent variables k, s, δ, we have[5]

$$d\mu = dk + \frac{\partial l}{\partial s}ds + \frac{\partial l}{\partial \delta}d\delta \tag{28}$$

$$\begin{aligned} dv &= -\frac{r_0}{(k+l)^2}\left[(1+\delta)(dk + \frac{\partial l}{\partial s}ds + \frac{\partial l}{\partial \delta}d\delta) - (k+l)d\delta\right] \\ &= -\frac{v}{\mu(1+\delta)}\left[(1+\delta)(dk + \frac{\partial l}{\partial s}ds + \frac{\partial l}{\partial \delta}d\delta) - \mu d\delta\right]. \end{aligned} \tag{29}$$

---

intervals corresponding to the stages of production swallowed by vertical combination. The reasoning of the whole §11 concerning the function $l(\delta)$ can be easily applied to $k(\delta)$. In §12 (including Fig. 1) and the following, the substitution of the function $k(\delta)$ for the constant k can also be performed without difficulties.

[5] Cf. however the preceding footnote.

By substituting this expression in (25a) we get

$$(30)\qquad \frac{d\mu}{dv} = -\frac{\mu}{v}\left[1 - \frac{\mu}{\mu - (1+\delta)\left(\frac{dk}{d\delta} + \frac{\partial l}{\partial s}\cdot\frac{\partial s}{d\delta} + \frac{\partial l}{\partial \delta}\right)}\right].$$

We now introduce the symbol E, defined as

$$(31)\qquad E = \frac{dk}{d\delta} + \frac{\partial l}{\partial s}\cdot\frac{ds}{d\delta} + \frac{\partial l}{\partial \delta} = \frac{d\mu}{d\delta} \text{ (cf. (28)).}$$

From (25) we have

$$E\cdot\frac{1+\delta}{\mu} = 1 - \frac{dv}{d\delta}\cdot\frac{1+\delta}{v}$$

and since

$$\frac{dv}{d\delta} > 0, \qquad \frac{1+\delta}{v} > 0$$

it follows that

$$(32)\qquad E < \frac{\mu}{1+\delta}.$$

Instead of (30) we can now write:

$$\frac{d\mu}{dv} = -\frac{\mu}{v}\left[1 - \frac{\mu}{\mu - (1+\delta)E}\right] = -\frac{\mu}{v}\cdot A$$

where

$$(33)\qquad A = 1 - \frac{\mu}{\mu - (1+\delta)E}$$

but here, from (32) we know only the lower half of the hyperbola is to be taken into consideration. Then

$$(34)\qquad \left(\frac{dp}{dv}\right)_{m=\text{const.}} = -\frac{m}{\mu^2}\cdot\frac{d\mu}{dv} = \frac{p}{v}\cdot A.$$

A has the following values:

If:	it follows:	Movement of prices with increasing v and constant m
1. $E = \pm\infty$	$A = 1$	increase proportionally
2. $E < 0$	$0 < A < 1$	increase less than proportionately

3. $E = 0$	$A = 0$	unchanged
4. $\frac{\mu}{1+\delta} > E > 0$	$A < 0$	decrease

Where $\delta = \text{const.}$ we have an example of Case 1, which we have already discussed in (27). Obviously, for the determination of μ the lower limit (24) must also be considered. Case 2 occurs in a situation, among others, when the change of the degree of differentiation influences the real stock of money less strongly than a simultaneous opposite change of the fluctuations in the stocks of goods, independent of the differentiation. Case 3 is referred to when it is said, in the sense of the "narrower" quantity theory, that the velocity of circulation has no effect on prices; we see then this can occur either in the case of an accidental compensation of the effects of the three independent factors considered k, s, δ, or in the case where $k = \text{const.}$, $s = \text{const.}$, $\partial l/\partial\delta = 0$. The condition $\partial l/\partial\delta = 0$ means, however, that the process of differentiation does not disturb at all the synchronization of arrangements for carrying inventories, which, naturally, is conceivable only as a limiting case.

In Case 4 the relation is the reverse of that in Case 2. As an example, consider the case where $k = \text{const.}$, $s = \text{const.}$ so that $E = dl/d\delta$ (and thus $E > 0$). We would have in this case a *fall* of prices brought about by the increase of the velocity of circulation.

§14. The "*self-acceleration*" of an inflationary or deflationary price-movement, due to automatic changes in the velocity of circulation of money ("flight into commodities" or "black-jack-game" in times of rapidly increasing prices, "flight into money" or "hoarding" in times of deflation), may be expressed in terms of a functional relation between μ (real money stock) and $\dot{p} = dp/dt$ ("price-tendency"), t representing time, and p being the price of a productive factor unit. In so formulating, we neglect, as we did in the preceding paragraphs, the various lags (between incomes and prices on different consumption-distances) which may, and, in fact, do, occur in times of price-shifting. Under the influence of a vigorous positive price tendency ($\dot{p} > \gamma_1$, γ_1 being a constant positive), the owners of full-paid commodity-stocks in the consumption-distance-interval $_{(x+\Delta x)}|\text{——}|_x$ tend to reduce the average time-distance between a selling out and the corresponding replenishing of their full-paid stocks; the real value, per time unit, of their payments or receipts, averaged for a time period, embracing a longer series of stock-oscillations, may, however, remain unchanged, as the volume of production remains constant. $r(x)$ and $r(x+\Delta x)$ will then remain constant, whereas the time elapsing between the time-point of minimal stocks (say, $t=0$) and the time point of maximal stocks (say, $t=\theta_x$, according to §8), will be reduced, say, from θ_x to, say, $_1\theta_x$, the real value

of maximum stocks will correspondingly (if not necessarily proportionally) decrease from $\overline{K}(x)$ to, say, ${}_1\overline{K}(x)$, and the new, reduced real value of maximum sales will be

$$ {}_1\bar{r}(x) = \lim_{\Delta x \to 0} \frac{{}_1\overline{K}(x)}{\Delta x} < \bar{r}(x). $$

The real money stock shown in the balance-sheets of the commodity owners situated in the consumption-distance-interval $_{(x+\Delta x)}|\!-\!-\!-\!-|_x$ will decrease from

$$ \mu_x = \int_x^{x+\Delta x} [\bar{r}(x) - r(x)]dx $$

to

$$ {}_1\mu_x = \int_x^{x+\Delta x} [{}_1\bar{r}(x) - r(x)]dx, $$

$$ \therefore \quad \Delta\mu_x = {}_1\mu_x - \mu_x = \int_x^{x+\Delta x} [{}_1\bar{r}(x) - \bar{r}(x)]dx. $$

The *total real* money stock $\mu = \int_0^\infty [\bar{r}(x) - r(x)]dx$ will, however, obviously change in the same direction as μ_x only on the condition, that

$$ \int_0^\infty {}_1\bar{r}(x)dx > \int_0^\infty \bar{r}(x)dx. \tag{35} $$

This condition is *not* fulfilled if, for example, only the consumers (stock-owners with lower values of x) are "flying into the commodity," whereas the producers (or, generally, stock-owners with higher x-values) do not change their previous attitude, and neither buy (and pay for) the raw materials, nor pay their workers in smaller time-intervals than before, but accumulate consumers' money notwithstanding the devaluation risk. Such a difference of behavior is, of course, only possible for short adaptation-periods. If the causes, which make $\dot{p} > \gamma_1$, continue to work for longer time, we must have $\Delta\mu_x < 0$ for all values of x, and $\Delta\mu = \sum_0^\infty \Delta\mu_x > 0$. In the same way we obtain $\Delta\mu > 0$ for the case of a negative price tendency $\dot{p} < -\gamma_2 (\gamma_2 > 0)$.

It is, furthermore, consistent with experience to assume that the incentive to reduce (or increase) cash in hand is the more vigorous and the more generally expanded among the mass of producers and consumers, the greater the absolute value of $\dot{p}$; or expressing μ as a function of $\dot{p}$, we should characterize $\mu = \mu(\dot{p})$ by the following statements

$$ \frac{d\mu}{d\dot{p}} < 0 \quad (-\gamma_2 > \dot{p} > \gamma_1;\ \gamma_1 > 0;\ \gamma_2 > 0). \tag{36} $$

$$ \lim_{\dot{p} \to \infty} \mu(\dot{p}) = 0; \qquad \lim_{\dot{p} \to \infty} \mu(\dot{p}) = +\infty. \tag{36a} $$

We may further assume that, as long as the price-movement lies within certain limits, the owners do not change their attitude at all, so that

(36b) $$\frac{d\mu}{d\dot{p}} = \text{const.} \quad (-\gamma_2 < \dot{p} < \gamma_1),$$ as represented in Figure 2.

Putting $\delta = \text{const.}$ the formula (36) can be transformed, owing to (26), in $dv/d\dot{p} > 0$, which is the usual expression of the mechanism of "hyper-inflation."

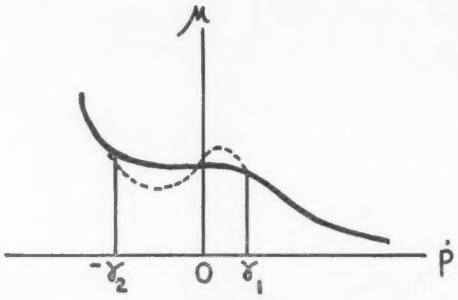

FIGURE 2

We have so far neglected the possibility of imperfect adaptation and, consequently, of the short-time lags between the price- (and income-) tendencies on different consumption-distances x. This neglect is, however, hardly allowed in the case of smaller values of $\dot{p}$, these values being, as we shall see, in certain conditions, peculiar to the starting stages of inflation (or deflation), and consequently correlated with the shortness of the adaptation-period. J. M. Keynes, for example, believes that in the beginning of inflation (or deflation) the movement of prices is likely to be in *positive* correlation with the real cash volume of the consumers μ_0, owing to the lagging of the consumers'-expenditure behind money-incomes. As long as this phenomenon is not compensated by opposite phenomena for higher values of x, the total μ itself would be expressed somewhat in the manner of the dotted line in Fig. 2. This hypothesis, as well as others, considering the lags between the prices or between the cash-holding policies of consumers and producers, or based on the technical fact of new money being preferably injected on higher consumption distances (producers'-credits prevailing relatively to consumers'-credits), ought to be proved empirically and could highly modify the curve $\mu(\dot{p})$.

It follows from $\mu = m/p$, $p = dp/dt$ and (36) that

$$\left(\frac{m}{p} - \frac{dm}{dp}\right): \frac{\frac{d^2p}{dt^2}}{\frac{dp}{dt}} > 0 \text{ or putting } \ddot{p} = \frac{d^2p}{dt^2}$$

(37) $$\left(\frac{m}{p} - \frac{dm}{dp}\right): \frac{\ddot{p}}{\dot{p}} > 0.$$

In the special case $m = \text{const.}$ (for instance in the case of a domestic price-shifting provoked by shifting of foreign prices or of exchange-rates), we have simply

$$\frac{\ddot{p}}{\dot{p}} > 0,$$

a self-acceleration of a price-tendency. In the more general case of changes in nominal money quantity provoking and (or) accompanying the price-changes (as for instance in the case of a budget-deficit generating inflation), the formula (37) shows that the price-tendency is self-accelerating only in the case of the rate of money-growing lagging behind the rate of price-growing.

In terms of §13 we should say that k and s, and consequently E and A, are functions of $\dot{p}$, which must be introduced in the equations (33) and (34).

V. APPLICATION TO TWO CONTROVERSIAL QUESTIONS

§15. The dispute between the "monetary" and "a-monetary" theorists on the question of velocity of circulation turns (according to Holtrop) on the fact that the "monetary" theorists (to which group Holtrop himself belongs) deny any connection between f and F or between v and V, while the "a-monetary" theorists seem to have believed in such a connection and moreover frequently in a relation of proportionality between these two quantities. Our equations show the real character of the connection (cf. §4):

$$F = \frac{r_0}{K}, \quad V = F(1 + \delta)$$

$$f = \frac{r_0}{\mu} = \frac{r_0}{\overline{K} - K + L - L_1}, \quad v = f(1 + \delta)$$

$$\frac{F}{f} = \frac{V}{v} = \frac{\mu}{K} = \frac{\overline{K} - K + L - L_1}{K}$$

$$= \frac{\int_0^\infty \bar{r}(x)dx - \int_0^\infty r(x)dx + L\left[\int_0^\infty \sigma(x)dx\right] - L_1\left[\int_0^\infty \sigma(x)dx\right]}{\int_0^\infty r(x)dx}$$

where $\sigma = \zeta(\delta, s)$. Here cf. the equations (13), (15), (15a), (21).

The circuit frequency (resp. velocity of circulation) of goods and that of money are thus not proportional to one another; they are related to one another, however, in so far as they are both dependent on the form of the function of manufacture $r(x)$; the circuit frequency (or velocity of circulation) of goods is, however, also affected by the functions:

L (dependence of liquidity-reserve L on σ)
L_1 (dependence of the decrease in stock of money, L_1, made possible by banking and insurance, on σ)
σ (dependence of the range of fluctuation, σ, of non-seasonal fluctuations in the stocks of goods, on the distance from consumption x)
ρ (dependence of the range of fluctuation, σ, of non-seasonal fluctuations in the stocks of goods, on δ and s.)
and by the value of:
δ (degree of differentiation)
and s (standard-deviation of non-seasonal fluctuations in the stocks of goods after elimination of the degree of differentiation.)

If we consider, for example, two economic systems, of which 1 possesses a greater capital equipment, relative to real income, than 2, then the relation of f_1/F_1 to f_2/F_2, if L, L_1, σ ρ, δ and s are equal in 1 and 2, depends only on the functions $r_1(x)$, $r_2(x)$, $\bar{r}_1(x)$, $\bar{r}_2(x)$, that is, on the functions of manufacture and on the seasonal fluctuations in the two countries.

§16. The question whether the velocity of circulation of money has an influence on prices or not (for the best statement of this controversy we are much indebted to A. Marget), is answered by the conclusions of Part IV. In particular, it is evident that the statement that every change in the velocity of circulation of money must be "neutralized by a proportional alteration of the velocity of circulation of goods" is, in general, *not* true. This view rests on the equation, itself correct,

$$mv = pVK$$

but which is interpreted as if $v/V = \text{const.}$ in all cases; thus, as if all other quantities were independent of v. In reality a change of $v = r_0/\mu(1+\delta)$ can be brought about by the variation of any one of the three factors r_0, μ, and δ, whilst in turn $\mu = k + l(s, \delta)$ [and at the same time $\mu \gtrless t/w + \beta$ cf. (24) $\cdots$] is dependent on δ:

(1) Only if total *real income* r_0 varies, when $\delta = \text{const.}$, $\mu = \text{const.}$ and $K = \text{const.}$ is $V/v = \text{const.}$ But this assumes the improbable case that real income r_0 varies without variation in the function of manufacture $r(x)$, which occurs only if the two economies to be compared owe the diversity of their real incomes to a difference in population alone, not to differing material equipments (state of the arts).

(2) If, on the other hand, the velocity of circulation of money varies because of a change in the *degree of differentiation* δ, where $r_0 = \text{const.}$, the real stock of money $\mu = k + l(s, \delta)$, where $k = \text{const.}$ and $s = \text{const.}$, will also vary so that $V/v = \mu/K$ can not remain constant; p moves in this case in a direction opposite to v, in a degree which depends on the numerical form of the function $l(\delta)$ or on the quantity w.

(3) Finally, if variation of v is brought about by a change in the *real stock of money* μ, which, in turn, is caused by variation, not of δ, but of k or s, that is, by variation in the character of fluctuations in the stocks of goods independent of the degree of differentiation, the assumption $V/v=\overline{K}-K+L-L_1/K=\text{const.}$, i.e., a strictly proportional variation of the stocks of goods K on the one hand and the sum $\overline{K}-K+L-L_1$, on the other, can hold good only by accident. The price movement is in Case (3) (i.e. where $\delta=\text{const.}$) in the same direction as the movement of v and proportional to it.[6]

The most important problem now seems to us to be the numerical analysis of the functions $r(x)$, $\bar{r}(x)$, $\sigma(x)$, L, and L_1; further, the determination of the influence of the degree of differentiation, i.e. the numerical analysis of the functions $l(\delta)$ and $k(\delta)$.

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[6] As to *empirical* investigations of the thesis of an alleged price neutrality of the velocity of circulation of money (Burgess, Snyder, O. Anderson), cf. A. Marget in the *Zeitschrift für Nationalökonomie*, IV, H. 2, pp. 199–200 and Marschak in the *Arch. f. Sozialw.*, Jan. 1933, pp. 409–412. I. Fisher is in Chapter VIII §5 of *The Purchasing Power of Money* the opponent of the thesis just mentioned. If, in spite of this, his book gave O. Anderson encouragement to verify this thesis statistically, this rests on a manner of expression perhaps somewhat lacking in precision (O. Anderson says, "energetic") in the closing sentence of Chapter VIII of Fisher's book. (Cf. O. Anderson, in *Zeitschrift für Nationalökonomie*, II, 535.)

NOTE ON ECONOMIC CYCLES AND RELAXATION-OSCILLATIONS

By L. Hamburger

In the July 1933 issue of Econometrica, M. Ph. Le Corbeiller published a communication "Les systèmes autoentretenus et les oscillations de relaxation," presented at the meeting of the Econometric Society in Lausanne, September 1931. In this communication he says:

Le problème des crises, et plus généralement des oscillations des prix, est assurément l'un des plus difficiles de l'Économie Politique; il ne sera sans doute pas de trop, pour approcher de sa solution, de la mise en commun de toutes les ressources de la théorie des oscillations et de la théorie économique. C'est pourquoi j'ai pensé pouvoir vous présenter un compte rendu succinct d'une avance récente, que je crois importante, de la théorie des oscillations: celle apportée au problème des systèmes autoentretenus par la découverte des oscillations de relaxation....

The present writer would like to point out that the applicability of the principle of relaxation-oscillations to economic cycles was first emphasized by him in 1928[1] in a discussion following a paper read by Messrs. Van der Pol and J. van der Mark on "The Heartbeat Considered as a Relaxation-Oscillation, and an Electrical Model of the Heart."[2]

This suggestion, at first somewhat ridiculed even by the pioneer in the field of relaxation phenomena, was subsequently corroborated by results indicated in my paper in *De Economist* LXXIX, pp. 1–38 (1930). A French version of this paper appeared January 1931 in *Supplément aux Indices du Mouvement des Affaires* IX, pp. 1–35 (1931).[3]

Perhaps it may be of interest to those who see economics as a part of natural science in general, and of biology in particular, to note that as early as January 9, 1922, the late Professor J. K. A. Wertheim Salomonson delivered a rectoral speech at the University of Amsterdam on the subject "Vital Actions Considered as Rhythmical Occurrences" (which has apparently entirely been overlooked subsequently), in which this forerunner emphasized the existence and importance in many phenomena of life of a kind of oscillation definitely differing from harmonic ones. He emphasized the very type which later found its mathematical determination and physical interpretation in the magnificent work of Dr. Balth. van der Pol.

[1] May 7, 1928, Meeting of the Batavian Society of Logic Empirical Philosophy.

[2] Cert. e.g. *Arch. Néerland. de Physiologi de l'Homme et des Animaux*, 3e livraison, p. 418 (1929).

[3] L. Hamburger "Analogie des Fluctuations Économiques et des Oscillations de Relaxation."

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FRANCIS YSIDRO EDGEWORTH (1906, *Aet.* 61)
1845–1926

FRANCIS YSIDRO EDGEWORTH

By Arthur L. Bowley

Among the pioneers of econometrics, Francis Ysidro Edgeworth must be given a very high place. Especially important are his contributions with regard to the study of the theory of economics and the theory of statistics by the help of mathematics. For actual measurement he would give place to Jevons, for the theory of measurement to no economist.

With the recent publication of J. M. Keynes's *Essays in Biography*,[1] the facts of Edgeworth's life are familiar. He was born in 1845 at Edgeworthstown, County Longford, Ireland, on the estate where his ancestors established themselves in the sixteenth century—reduced to very small value by the time he inherited it in 1911. His father (1809–1847), who appears to have spent his life in the peripatetic pursuit of knowledge, is described by Thomas Carlyle[2] as he appeared about the year 1836; Maria Edgeworth (1767–1847), the celebrated authoress, well-known in the literary circles of the early nineteenth century, was his aunt. His mother was a Spanish refugee, married after a brief acquaintance in London. One of his great-great-grandfathers was a Huguenot refugee. Francis Ysidro was the youngest of five sons and survived all his brothers. His features suggested his Spanish descent; in his knowledge of French, German, Italian, and Spanish, and in his ready acquaintance with economists of all nations, there is a suggestion of his international origin.

In 1862 he entered at Trinity College, Dublin, where it may be presumed he studied both mathematics and classics. He was never at a loss for a classical quotation. I have found no record of his graduation at Dublin, but in 1867, in his twenty-second year, he went to Oxford, and in 1868 was admitted to Balliol College. In 1869 he was awarded first class honors in Literis Humanioribus, the great school of Philosophy; but he did not actually take his degree (B.A.) till 1873.[3] He was admitted to the Bar in 1877. During the ten years after he reached Oxford his interests appear to have been philosophy, ethics and, subsequently, economics. He was immersed in the writings of Bentham and others of the utilitarian school, while in economics he was educated in the works of John Stuart Mill. In this period he probably neglected

[1] Keynes's study of Edgeworth is essentially the same as the obituary that appeared immediately after his death, *Economic Journal*, 1926.

[2] *Life of Sterling*, Part II, Chapter IV.

[3] Subsequently he took the M.A. degree in 1877 (a matter of formality), and at a later date was given an honorary D.C.L. of the University of Durham.

mathematics, for his mathematical writing indicates a want of systematic training. Though he shows great insight into the principles of mathematics, there is a want of facility and neatness in his handling of problems. Familiar as he was with the work of Laplace, Todhunter, and Clerk Maxwell, he had difficulty in elementary applications. His line of thought is often a little obscure; sometimes he labors the obvious, and at others is so brief as to be difficult to follow. He was always the victim of numerical mistakes and errors in writing and printing. Apparently he settled in London immediately after he left Oxford, and had no definite occupation.

PHILOSOPHY AND THE PHILOSOPHICAL ASPECT OF ECONOMICS

It is evident that in the first part of his life he turned his mind principally to philosophic questions and especially to the relation between ethics and economics. His first known publication is a paper on "New and Old Methods of Ethics" in 1877. He began to interpret utilitarian and economic ideas by mathematical symbols, and very early he must have realized the importance of the conception of probability. The result of this period of study and incubation was the publication of his *Mathematical Psychics* in 1881. The mathematical economics in this book he owes primarily to Cournot, Jevons, and Gossen, something to his great contemporary Marshall;[4] the ethics is developed from Mill and Sidgwick; but the whole conception and treatment are original in the highest degree. It was so original that its importance has been only very gradually realized. The circulation was limited and it was little known;[5] the number of philosophers or economists qualified to understand it was very small, but among them were Jevons and Marshall. The following extracts from *Memorials of Alfred Marshall*, edited by A. C. Pigou, are interesting from many points of view.[6]

> Alfred Marshall first became for me a notable name when Jevons [in 1879 or 1880], conversing about mathematical economics, recommended as the latest contribution to that subject the now celebrated papers on the *Pure Theory of Foreign Trade and Domestic Values*. At the same time Jevons highly praised the then recently published *Economics of Industry*. Eagerly studying these writings, I discerned a new power of mathematical reasoning, not only in the Papers

[4] Cournot 1801–77, Gossen 1810–58, Jevons 1835–82, Marshall 1842–1924, Edgeworth 1845–1926, Wicksell 1851–1926.

[5] "Three days after Part II was finished, I received and saw for the first time the *Mathematical Psychics* of Professor Edgeworth." Irving Fisher in Preface to *Mathematical Researches in the Theory of Value and Prices*, dated 1892. (The phrase is re-translated from the French edition.)

[6] P. 66, *Reminiscences* by Professor F. Y. Edgeworth.

bristling with curves and symbols, but also in certain portions of the seemingly simple textbook. With reference to such passages, writing in the year 1881, I characterized the author by a phrase which he himself afterwards acknowledged to be appropriate, "bearing under the garb of literature the armour of mathematics." The phrase might be applied to many passages in the text of the *Principles of Economics*.

J. M. Keynes on Marshall (pp. 25–26):

In 1881 [Marshall], reviewing Edgeworth's *Mathematical Psychics*, after beginning "This book shows clear signs of genius, and is a promise of great things to come," adds "It will be interesting, in particular, to see how far he succeeds in preventing his mathematics from running away with him, and carrying him out of sight of the actual facts of economics."

Perhaps Marshall did not appreciate the path-breaking quality of the book, since he was already averse from the mathematical exposition of economics. The reading of it is at least essential to all who wish to understand the concepts that underly Edgeworth's later writings, and it is fortunately now easily accessible, since it has been reprinted at the London School of Economics (price 5s.).

PROBABILITY AND STATISTICS

We next find Edgeworth appointed as Lecturer in Logic at King's College, London, in 1880; in 1890 he succeeded Thorold Rogers there as Tooke Professor of Economic Science and Statistics, a chair more noticeable for the distinguished men who have held it than for its emoluments.[7]

Now begins the period of the publications on the theory of probability. In 1883–84, at least six papers appeared in this sphere, of which the first was "The Law of Error" (*Phil. Mag.* 1883). While *Mathematical Psychics* laid the foundation and showed part of the construction of the edifice of his economic theory, the paper (only 36 pages) on *Methods of Statistics*, read at the Jubilee Meeting of the Statistical Society in 1885, exhibits at once the whole plan of his statistical work. To use the kind of metaphor in which he delighted, the foundation, based on the work of Laplace, Lexis, and Venn, was laid, the first story was completed, and the frame work of the second was partly set up, partly in the process of execution. In the next forty years more stories were added, additional buttresses were erected where the structure was weak, and innumerable decorations within and without added to the amenities of the edifice in the best classical and italianate styles, while secret rooms were provided for initiates. There is little in the

[7] The professors have been Thorold Rogers, Edgeworth, Cunningham, Urwick. After the war its annual value was about £50, and it was merged in another professorship, now held by F. A. von Hayek under the original title.

subsequent work whose origin cannot be traced through this paper, but, as with *Mathematical Psychics*, it was appreciated by very few, and the development and the use of the important principles which Edgeworth was the first to introduce to English statisticians was left almost entirely to him for many years.

Edgeworth was the philosopher of statistics rather than the practitioner. We may give prominence to two of his main subjects, those in which his point of view differs from most modern statisticians.—Every judgment based on mathematical chance is related to *a priori* probability. There must be some presumption about the field in which the events take place. If an event occurs of which the chance is small, e.g., five successive throws of double-six with a pair of dice, we must choose between the alternatives that the dice were loaded and that an improbable event has taken place. If the difference between the average of two samples is greater than the three times the computed standard deviation, we have the alternatives that the samples were badly selected, that an unusual event has taken place, or that the populations from which they were drawn were essentially different. Or again we may need to assume, as in field samples in agricultural tests, the existence of an underlying normal curve of variation. Edgeworth frequently used the principle of maximum probability and deviations from it—his method of "genuine inverse probability"—but his treatment of it was essentially different from that of Professor R. A. Fisher. I understand the latter to deny the applicability of the conception of *a priori* probability and to make strenuous efforts to evade its use. Edgeworth was convinced that there was always an element of the unknown, that in the end we must fall back on unmeasured experience. But in his hands this did not mean that we could not obtain useful results, for in appropriate cases the influence of the unknown could be reduced to such small dimensions as to leave the result almost unaffected on any reasonable hypothesis. It is not certain, however, that this judgment, based principally on economic statistics, would be applicable to all physical experiments, though he traced it in the theory of atomic motion.

The second subject which Edgeworth developed from a specially distinctive view is the Law of Error. His Law of Great Numbers is usually erroneously classed as one of a species which contains the formulae of Thiele, Charlier, Karl Pearson, and others. It is true that the mathematical expression is very similar, but the fundamental conception is not the appropriateness for representing experimental results, but the determination of a form that would be derived from given hypotheses; the experimental verification is secondary. The hypotheses, based on Laplace's work, are the existence of numerous independent causes and their interplay in producing aggregates or

averages. According to the number and strength of these causes, the successive terms of his law of great numbers have greater or less importance. This law is

$$z = e^{-(-1/3!k_1D^3+1/4!k_2D^4-)}\ (e^{-x^2/c^2}/c\sqrt{\pi})$$

where z is the frequency with which a deviation of x from the average of measurements occurs, and c, k_1, k_2, are constants depending on the successive moments of the curve. Qualifications are introduced when the causes are not completely independent and when successive experiments are inter-related. A development to the 'method of translation' is made, when the observed quantity is not itself distributed according to the law but is some function of a variable so distributed. The formula is extended also to two or more variables.

It was a favorite theme for Edgeworth that the normal law of error, or its generalized expression, is prevalent in nature and in the subject matter of economics, however much disguised; or, if not in the raw material, then in the play of averages. It is hardly yet realized how nearly normal is the distribution of averages, even when the number of their constituents is small. The law of small numbers is practically indistinguishable from the normal law, unless the numbers are indeed very small. This was pointed out very clearly by Edgeworth. He said, "if the Greeks had been acquainted with the Law of Error, they would have erected an altar to it," presumably alongside that to the Unknown God. His interest was rather in the fundamental prevalence of the law than in its applicability to the representation of groups; his long series of papers on the Mathematical Representation of Statistical Data (*Statistical Journal*, 1913–18) was, I think, more valuable for its by-products than for its major thesis, and Edgeworth himself expressed doubt of their use. The important applications were in measuring the accuracy of averages, and the significance of the differences between them. His study of fluctuations (*Statistical Journal*, 1885) may be compared with Professor R. A. Fisher's method of variance; the one in a cross table of death-rates in years and districts, the other in a Roman square of agricultural experiments.

A general view of the several facets of Edgeworth's intellectual interests is obtained by studying his work on Index-Numbers of Prices, beginning with his reports to a British Association Committee, 1887–1890. On the statistical side, we find the examination of the merits of various types of averages, of the effect of weights and their small importance in this problem, and of the application of the law of error for determination of precision.[8] On the economic side, there is a very care-

[8] "By rejecting the Calculus of Probabilities (Mr. Walsh) has . . . thrown away an instrument necessary for the performance of that measurement" (i.e. of the value of money). *Papers*, II, 376. (Written in 1901.)

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ful and systematic analysis of the quaesitum, the quantity to be measured for particular economic purposes, and of the relation to currency problems. "Beneath the apparent unity of a single question there is discoverable upon a close view a plurality of distinct problems."[9] The logical basis of the investigation is further developed in later papers, especially those which deal with Mr. Correa Walsh's work. It may be recommended to anyone who wishes to do serious work on Index Numbers to read Professor Irving Fisher's *The Making of Index Numbers*, Mr. Walsh's *The Problem of Estimation*, and Edgeworth's series of papers, reports, and reviews. If, in consequence of inability to bring all the views expressed into a coherent body of thought, our aspirant gives up the problem, he will at least have obtained an insight into the psychology of the writers.

In 1892 Edgeworth published his first paper on Correlation. It is idle to try to determine whether he or Professor Karl Pearson can claim priority in arriving at the main ideas which have led to the now well-known methods. Both attacked the problem of joint variation at the point which Galton had reached. Edgeworth gave in 1892 the general formula for multiple correlation in essentially the same form as is now used, viz.,

$$z = Ce^{-1/D.(x_1^2 R_{11} + x_2^2 R_{22} + \cdots + 2x_1 x_2 R_{12} + \cdots)}$$

where R_{11}, R_{12} $\cdots$ are the minors of the determinant

$$D = \begin{vmatrix} 1 & \rho_{12} & \rho_{13} & \cdot\cdot \\ \rho_{12} & 1 & \rho_{23} & \cdot\cdot \\ \cdot & \cdot & \cdot & \cdot\cdot \\ \cdot & \cdot & \cdot & \cdot\cdot \\ \cdot & \cdot & \cdot & \cdot\cdot \end{vmatrix}.$$

Here ρ_{12} is an average of the inverse ratios of an x_1, to that value of x_2 which is most frequently associated with it. This average was more closely investigated and defined in 1893, and an expression for it was obtained which was subsequently identified with Pearson's sum-product formula.

The numerous statistical studies published between 1893 and 1926 are to a very large extent the working out of ideas expressed or latent in the papers already named, with numerous applications to a great variety of problems and with critical and explanatory references to the work of other writers. Throughout the twoscore papers listed for these years runs the thread of the importance of sound fundamental ideas on probability in all mathematical statistics as opposed to purely empirical work. The ground deliberately chosen as central to his posi-

[9] *Papers*, I, 199.

tion is described in his Presidential Address to the Royal Statistical Society in 1912 "On the Use of the Theory of Probabilities in Statistics Relating to Society."

In writing of Edgeworth's attitude to statistics I feel that I am on fairly safe ground. When, in 1895, I was appointed to lecture on Statistics at the newly founded School of Economics, on Marshall's introduction I wrote to him for advice on the nature and literature of the subject, and he recommended principally Venn's *Logic of Chance*, Todhunter's *History of Probability* and Lexis' *Zur Theorie der Massenerscheinungen*, to which I naturally added his 1885 paper at the Jubilee meeting and his reports on Index-Numbers. From that time till his death I constantly learned from him, worked with him, and met him frequently in London and Oxford. It was with difficulty that I could turn the conversation from the nature of probabilities and the applications of the Law of Error. A trivial anecdote illustrates this. In 1904, a party of economists was bicycling out of Cambridge, and, with some danger to the traffic, Edgeworth began to discuss the method of translation or some similar topic; Professor Cannon drew up alongside and said, "Put on the pace, Bowley, he can't talk mathematics at more than 12 miles an hour."

GENERAL ECONOMICS

In Economics proper, to which I now turn, I cannot claim the same intimacy, and depend rather on the judgment of economists as shown in their quotations from his works, and especially on Pigou's very sympathetic review of Edgeworth's collected papers in the *Economic Journal*, June 1925.

Wherever economic theory called for mathematical treatment Edgeworth's interest was specially aroused, and, though this test by no means admits all the topics he treated, it is of special importance to econometricians.

It is the original analysis in *Mathematical Psychics* that has found the principal place in economic literature, namely the invention or discovery of Indifference Curves and their relation to the Contract Curve. The same analysis was applied in 1893 to the Pure Theory of International Values, and the findings of this study have been incorporated widely in treatment by later authors. Edgeworth did not seek after exceptions or paradoxes for their own sake; but in conscientious analysis he found that accepted rules were only true approximately and within limits. Thus, though convinced that free-trade was the best policy, at least for England, his analysis led to determination of the cases where an import or export duty was in part borne by the foreigner. He was greatly interested in Mr. Bickerdike's view of "in-

cipient taxes" which were advocated as benefiting an importing country, but his final judgment was adverse, as is seen in the concluding sentences of his criticism, which may be quoted in full as typical of his style and outlook.

> Thus, the direct use of the theory is likely to be small. But it is to be feared that its abuse will be considerable. It affords to unscrupulous advocates of vulgar Protection a peculiarly specious pretext for introducing the thin edge of the fiscal wedge. Mr. Bickerdike may be compared to a scientist who, by a new analysis, has discovered that strychnine may be administered in small doses with prospect of advantage in one or two more cases than was previously known; the result of this discovery may be to render the drug more easily procurable by those whose intention, or at least whose practice, is not medicinal. It was thus that the "drama of poison" perpetrated in the reign of Louis XIV was initiated by one whose baleful receipt was obtained from Glaser, a chemist of eminence, the discoverer of a new substance. Let us admire the skill of the analyist, but label the subject of his investigation *Poison*.[10]

From Cournot onwards, it has been recognized that the theory of monopoly demands essentially mathematical treatment. In this connection we find numerous studies relating to taxation, to railway rates, and other topics. The possible advantages of monopoly, in every stage from its establishment in one industry to complete socialism, are brought to light, in some cases to be condemned, in others accepted. In particular, Edgeworth discusses the possibility of discrimination, that is, rates or charges differentiated by place or class of customer, which is not inherent in pure competition but may result in net benefit to all concerned. There is very much that is important in these studies besides the often-quoted proof that in a specially devised and exceptional case a tax on one of two rival commodities may result in a lowering of the price of both. The mathematical results need for their application statistical data, such as are the proper study of econometricians—for example, questions on whether certain functions are positive or negative—and depend on the actual measurement of their elasticity.

We may agree with Pigou that the papers on the "Application of Probabilities to Economics" (*Economic Journal*, 1910, pp. 286 and 441 *seq.*) are most noticeable, and we may add, most characteristic. One of the earlier sentences in the first paper is "The theory of Probabilities lends to Economics, as to other sciences, certain premises which are evidenced, neither by pure intuition nor by formal induction, but by general impressions and what may be called mathematical common sense." We can often proceed a considerable distance with certainty by the help of accepted postulates, such that demand in

[10] *Papers*, II, 365.

general falls with an increase of price. Presently the results depend on the neglect of quantities which may be presumed to be small; such results are probably true in the light of general experience of the behavior of continuous functions. Though no numerical measurement of chance can be obtained, it may be affirmed that the probability of the failure of a theorem is very small. We may even go further and assume that in the absence of specific information a positive value of a variable is as likely to occur as a negative value, and even this meagre datum may afford definite guidance. It is to be remarked that in these papers are included not only instances of mathematical chance, but a number of applications of the conception of continuity of functions, where arguments are commonly based on unverified assumptions, not proved to be valid, but held to be reasonable or probable. Closely allied to this is the neglect of quantities presumed to be small.

Though there seems to be a cleavage between the studies on the theory of probability and those on international trade and monopoly, to Edgeworth there was an underlying unity in the fundamental conceptions and in much of the method of Economics and of Statistics; this unity found its expression in the mathematical treatment of both, and his interest was most easily aroused where mathematical ideas were involved, whether in ethics, economics, or statistics. It is in this region of fundamental similarities that we find what is most characteristic of his genius, that which distinguished him from almost all other economists and statisticians.

Neither subjective happiness nor belief can be measured, but indicators can be found in allied measurable quantities. Wealth has a relationship to welfare, belief can be connected with mathematical probability. There are scales of greater and less for incommensurable objects. The utilitarian's ideal of the greatest possible happiness is related to the mathematical economic conception of maximum utility. The best judgment or belief to be obtained from statistical data can be founded on algebraic maxima. From the fundamental expression for maximum utility and maximum probability are derived the detailed equations of exchange on the one side, and such formulae as those for 'least squares' on the other, as in Mechanics the principle of minimum potential energy leads to equations of equilibrium. While the foundations are similar, there is also interaction in the developments. Utilitarian ideas are involved in the choice of the 'best mean,' and probability justifies the conception of the representative firm, of the average man, and of the similarity of groups in relation to utility. In particular in many cases, where the solution of an economic problem depends on the determination of the direction of a curve, *a priori* probability will supply the most plausible answer.

EDITORIAL WORK

In 1891 Edgeworth succeeded Thorold Rogers[11] as Drummond Professor of Political Economy at Oxford and held that chair till he resigned in 1922 and became Emeritus Professor. During the whole of this period and till his death in 1926 he was Editor, or one of the Editors, of the *Economic Journal*, the first issue of which was in March 1891.

So far as he had a home it was, according to the time of year, two rooms at Mount Vernon, Hampstead, on the outskirts of and overlooking London, or in his Fellow's room at All Souls College, Oxford. Characteristically he had his own means of transit; from Oxford he bicycled by a route known to his friends as "Edgeworth's way" 30 miles to Great Missenden, whence he took train to Hampstead. Very few of his friends ever penetrated to his sanctum at Oxford—bachelor's rooms, for he never married—very many were entertained by him at All Souls College and at the Savile Club, London. He made no collection of books. In London he was to be found working in the rooms of the Statistical Society, and there or at Oxford he studied the books sent to the *Economic Journal* to review. He was also to be met at the sessions of the Statistical Society (President in 1912–14) at the Economics Society (that used to meet at University College, London where H. S. Foxwell was professor), and at the historic Political Economy Club, to which he was elected in 1891;[12] these three societies meet in regular sequence in the first three weeks of each winter month. He was for many years regular in attendance at the annual meetings of the British Association for the Advancement of Science, being President of the Economic Section in 1889. His name was in the first list of elected members of the British Academy in 1903. Edgeworth was thus the most accessible of the English economists. He usually spent his vacations in the Alps or near Dublin, but otherwise was not a great traveller.

The professional duties at Oxford were during his tenure of office very light. Economics till after his retirement was not a major subject in any 'School,' and in any case the custom at Oxford is to depend

[11] Thorold Rogers was Professor at Oxford 1862–68 and again from 1888 till his death in 1890. He was Tooke Professor in London from 1859 till 1890, occupying the two chairs simultaneously. Edgeworth gave up the Tooke Professorship on his appointment at Oxford. He was succeeded in London by Dr. W. Cunningham, and at Oxford by D. H. Macgregor in 1922.

[12] The first paper he read there was entitled "Under what conditions, if any, is the burden of a customs duty not borne by the consumers of the imported commodity," in 1893. The date is important in connection with his later work on tariff problems.

rather on College lecturers and tutors than on professorial teaching. Though he had great influence on many individuals, he did not organize any corporate teaching; the few courses of lectures he gave were not well attended, for indeed he had no faculty for that method of teaching. His work was in fact almost independent of his position at Oxford.

Mr. Keynes has testified to the great importance of his editorial work. It was specially marked in the organization of reviews of books. Though Edgeworth's direct writings turn on a limited number of subjects, his acquaintance with economic theory and with economists was very extensive in place and time and his memory was remarkable. In the reviews written by himself and republished, his encyclopaedic knowledge is as evident as the acuteness of his critical powers; but even these reviews are selective, for he appears to have looked critically at every book that reached the *Journal's* office. A request for a review would be accompanied by some apposite remarks on particular points in the text. Because he never developed a system of economics and never published a book except *Mathematical Psychics*, there is an erroneous impression that his sole interest was in refinements and exceptions, in mathematical curiosa rather than in the broad stream of economic thought; this impression is completely dispelled by reading his volume of reviews. Since nearly the whole edition of his *Papers* has been sold, it is to be hoped that a just view of his range has now been reached by all competent judges.

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Readers of Edgeworth's writings[13] are often deterred by what appears to be deliberate obscurantism in the arguments. Many by-paths are followed and left with a quotation from the classics; mathematical curiosa are interspersed with poetry; elaborate metaphors are developed, where one would expect rigid deductions. But if one studies carefully a treatise as a whole, with some knowledge of Edgeworth's general lines of thought, and then reads it a second time, one finds that the whole is coherent, the arguments valid and consecutive, the theme

[13] A nearly complete study of Edgeworth's work can be made from the sources here listed:

Papers Relating to Political Economy. Three volumes. Published on behalf of the Royal Economic Society by Macmillan and Co., 1925.

Review of the above by A. C. Pigou, *Economic Journal*, June 1925.

Mathematical Psychics, 1881, reprinted by the London School of Economics, 1932.

F. Y. Edgeworth's Contributions to Mathematical Statistics. Published as a separate pamphlet by the Royal Statistical Society, 1928. This contains an annotated bibliography of seventy-four papers by him on mathematical statistics.

is made clearer and more vivid by the variations. "By steps that are neither violently abrupt nor tediously circuitous, he reaches the heights from which the mutual dependence of all economic quantities can best be contemplated. At those heights too, are observed some curiosities of theory, like Alpine flowers, found only at great altitudes." These words may surely be transferred from his review of a book by another writer and applied tc himself.

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SIGNIFICANT DEVELOPMENTS IN BUSINESS CYCLE THEORY

By Felice Vinci

The theory of business cycles is that part of economic dynamics which is the most susceptible of scientific progress, as it may be founded on a mass of quantitative data generally allowing analytical formulations and experimental verifications.

In my opinion, the latest and most consistent progress in this field is due to Irving Fisher, who, in his last volume, *Booms and Depressions*, collected the results obtained by him and proposed a programme of most fruitful researches.

The readers of this journal are already acquainted with the subject dealt with in the above-mentioned work, as the author himself summarises it in the October 1933 issue.

In 49 propositions, illustrated by eight charts, Fisher first of all showed that there are many factors in business cycles, thus following Pigou, Mitchell, and other authors, adding that the three greatest cycles of the last one hundred years (culminating in 1837, 1873, 1929) were chiefly caused by over-indebtedness, impelled either by actual enlargements in the fields of investment of capital, or by developments of the technique of production or of conveyances, or by new inventions and discoveries, or by post-war reconstructions. The endeavor to liquidate that enormous mass of debts (generally after an alarm) caused a considerable fall of prices and an exceptional revaluation of money, which made debts still heavier and made depression worse; so that there were and there could be no other issues but bankruptcy or the policy of raising the level of prices.

Its process should logically be the following: *Debt liquidation* causes *distress selling*, which is followed by *contraction of deposit currency* and by a lowering of its velocity of circulation; the *level of prices* is lowered; the *net worth of business* is still more reduced; *profits and output* fall, as well as the employment of labor; *confidence* fails and *hoarding* increases; at last there will be a *fall in the nominal* or *money rates* and a rise in the real, or commodity rates, of interest.

These factors, however, not only form a *chain* of causation, but are connected by interrelations, which Fisher describes approximately (page 343 of the above-mentioned paper and Appendix I of the book) and to which he calls students' attention.

It is a whole field of research that he puts before the students of business cycles, showing how some of these relations were already searched into by him with remarkable results. He dwells chiefly on the relations

between the monthly fluctuations of prices and the employment of labor, the chart of which he brings up to date (charts II and III in the October 1933 issue of ECONOMETRICA, pp. 352, 353) and which, being confirmed by over thirty years' data (1903–33) really deserve being closely and carefully examined and being represented and explained in a suitable analytical form.

Let us say beforehand that it has been long noticed that high prices do not particularly stimulate the volume of business and the employment of labor, nor do low prices depress business. It is rather the *rise* or *fall* in prices that exert an influence. Such rises and falls generally appear first in the prices of commodities, and they but slowly get to wages, interest, and incomes; therefore they generally give profits to entrepreneurs in periods of rises and losses in periods of declines.

Those rises and falls, however, do not develop all their effects at once, but exert a gradual influence; and it is probable that the law governing this spreading should vary from one time to another as well as from one place to another, according to the particular index of prices by which rises and falls are measured and also according to the degree of sensibility of the producing classes.

Fisher,[1] considering this, had already estimated for the following months of the years 1916–22 the monthly percentage fluctuations in the official index of wholesale prices calculated by the United States Bureau of Labor, and taking for granted that every monthly fluctuation should exercise an influence of about 3 per cent on the trade of the following month, of about 6 per cent on that of the third month, of about 7 per cent on that of the fourth and fifth month, and that it should then slowly diminish its effects according to a fixed law, he built the curve resulting from these collected effects: that is what might be called the *hereditary curve of prices*.[2]

By using an index of trade calculated by Warren M. Persons for the United States for the same years and elaborated in order to remove the so-called secular trend and seasonal variations, he compared the two lines at a parity of standard deviation, thus getting a strong parallelism, which was disturbed only by irregular fluctuations and exceptional events, above all by the consequences of the Great War.

[1] I. Fisher: "Our Unstable Dollar and the So-called Business Cycle," in *Journal of the Amer. Stat. Assoc.*, June 1925.

[2] Fisher takes for granted that a given fluctuation should be spread in the following months so that, by representing the months on a logarithmic scale, we should get a Gaussian curve, whose arithmetical mean should be 9¼ months distant from the month to which the fluctuation relates. The Gaussian curve is indeed asymptotic in both directions, but this could not cause perceptible inconveniences to Fisher's applications.

He secured less satisfactory results from the data of the period 1877–1924—though on the whole they were not contrasting each other—as the hereditary distribution of effects is not constant in time and it is very difficult to reveal their variation. It is evidently not sufficient to show that the hereditary distribution used gives a strong parallelism of results: such parallelism going on for a long period is a necessary but not a sufficient criterion of its reliability. It is necessary also that the distribution should be admissible from the economic theoretical point of view. For instance, an always decreasing distribution or an always increasing one would be plainly absurd.

Considering the close connection between trade and employment of labor, it was to be expected that such connection should exist also between the monthly percentage *change* of prices and the indices of employment of labor. Indeed, Fisher, by making this comparison for the United States in the period 1903–1925,[3] got a close correlation, with the same exceptions as in the preceding case. Now, by extending these elaborations to 1933, he got a full confirmation of such results. They can be translated into an analytical form.[4] By calling $P(t)$ the index of prices at the time t, we may first of all say that:

$$\mu(t) = \frac{1}{P(t)} \frac{dP(t)}{dt} \tag{1}$$

is the instantaneous rate of variation in the index of prices.

By calling again $T(t)$ the index of trade (or of employment of labor), the parallelism ascertained by Fisher may be translated into the following equation:

$$\int_t^\infty p(t)T(t)dt = a + b\mu(t) \tag{2}$$

where a and b are constants, and $p(t)$ are weights obeying a certain hereditary law caused by the rate $\mu(t)$, the latter being theoretically supposed to extend its influence to infinity.

Generally, as it is not to be forgotten that Fisher's two series might have separate trends, we must write:

$$\int_t^\infty p(t)\,[T(t) - g(t)]dt = a + b[\mu(t) - \rho(t)] \tag{3}$$

[3] See *Revue Internationale du Travail*, June 1926.

[4] See F. Vinci in *Rivista Italiana di Statistica*, Bologna 1931, III, page 13 and page 123; and *Atti dell'Istituto Nazionale delle Assicurazioni*, Vol. IV, (Rome 1932); and *Manuale di Statistica* (Bologna, Zanichelli, 133), Part III, Chapter V, n. 2.

where $\varphi(t)$ and $\rho(t)$ represent the trends. The equation (3) can also be thrown into the form

$$\int_t^\infty p(t)T(t)dt = K(t) + b\mu(t). \tag{4}$$

The last equation may well be connected with analogous relations obtained on other phenomena. For instance, in June 1930[5] we pointed out for Italy a close correlation between the monthly percentage fluctuations of wholesale prices and the indices of share prices, which in our country seem remarkably trustworthy. It was confirmed in the following months, though in 1931 the general behavior of both series began to diverge (see Figure 1). By extending the analysis to 14 states and comparing the deviations obtained from observed data and smoothed ones, S. Vianelli[6] obtained the result that there is a strong tendency for relative deviations in share quotations to equal the relative deviations in the percentage fluctuations of prices, and that such tendency is only occasionally disturbed by alterations in currency and extraordinary events.

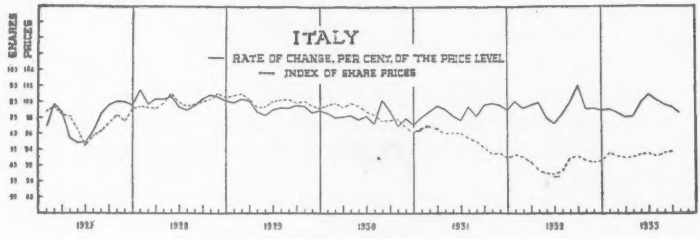

FIGURE 1

Such a result depends on the fact that month-to-month *changes* of prices are a more significant index of trade and profits than the *magnitude* of the level of prices computed with reference to a fixed base year. Furthermore, the indices of shares, through the influence of a great variety of actual conditions and psychological impulses, is really equivalent to the foreseen conditions of a wide section of the type of concern for which joint stock companies are characteristic representatives. The divergency of the trends depends on the rate of interest and on the tendency of the public to prefer deposits, etc., to stocks, or vice versa.

[5] *Rivista Italiana di Statistica*, 1930, II, p. 125 and chart 2.

[6] See: Silvio Vianelli: "Confronti Internazionali fra Prezzi delle Merci e Corsi delle Azioni," in *Rivista Italiana di Statistica, Economia e Finanza*, Dec. 1933.

The existence of such relations was already recognized in the old truism that the quotations of shares rise generally before the level of prices when the latter has not yet risen from the bottom, and they fall before the rising level of prices has reached the top; so that the Harvard Economic Committee concludes that stock exchanges give premonitory signs of the rise and fall of the level of prices.

This result may also be translated into an analytical form. By calling $C(t)$ the index of the quotations of shares at the time t, and by taking into consideration equation (1) we may write:

$$C(t) = a' + b'\mu(t) \tag{5}$$

and, generally, by letting $f(t)$ and $\rho(t)$ denote the trends

$$C(t) - f(t) = a' + b'[\mu(t) - \rho(t)] \tag{6}$$

$$C(t) = F(t) + b'\mu(t). \tag{7}$$

We deduce

$$\mu(t) = \alpha C(t) + R(t)$$

$$\text{g } P(t) = \int [\alpha C(t) + R(t)]dt + k$$

$$P(t) = P_0 e^{\int_0^t [\alpha C(t) + R(t)]\,dt}. \tag{8}$$

Taking for granted that in a certain interval the trends of the two series and the corresponding deviations may be assumed to have a certain analytical form, it will be possible in (8) to eliminate t and to obtain C as a function of P, this functional relation holding good within the limits of the chosen time interval.

Let us now compare this result with the preceding one. From (2) and (5) we get

$$C(t) = a'' + b'' \int_t^\infty p(t)T(t)dt \tag{9}$$

and from (3) and (6):

$$C(t) = W(t) + c \int_t^\infty p(t)T(t)dt. \tag{10}$$

Therefore, apart from the trend factor, the indices of quotations of shares appear as *proportional* to a cumulation of the future anticipated volume of business. This is quite reasonable. We believe that equations of the type (10) are going to play an important rôle in the study of economic dynamics. It will be necessary to go deeply into this subject and to extend the analysis along the lines drawn up by Fisher.

We believe that these tools will materially aid in the search for that

general theory of economic cycles that some students have vainly tried and are still trying to build up from purely abstract hypotheses.

Both the so-called "equation of exchange" and the so-called "Cambridge equations" or "the Keynes fundamental equations" can doubtless lead to interesting conclusions, but—as Fisher rightly understood—they cannot reveal the nature of the connections between the month-to-month *changes* in the quantities appearing in them. They are not truly dynamic equations. In order to arrive at such relations, we need a thoroughgoing quantitative analysis of facts and their "routine of change," and an expression of these facts in the form of equations. This is, after all, the true scientific analysis.

There are now great discussions and subtle reasonings about the above-mentioned equations and particularly about those of Keynes, but the contradictory results and the differences of opinion existing in the interpretation of these equations and in the difference between leading principles of monetary policy to which the equations lead, are a new evidence that the analysis was not sufficiently founded.

Let me add that, in my opinion, the discussions have not been carried on in a fruitful way, even with regard to the comparative merits of the equations. We do not believe—as Keynes states—that the "equation of exchange" should necessarily concern cash transactions and that the index of prices considered in it should be substantially different from that of the "Cambridge equation."

If we call M_1, M_2, M_3, the average number of monetary units circulating in a certain time and place either as *money* (metal, fiduciary, and token coins) or *substitutes* (checks, etc.) and *instruments of credit payable on maturity* (bills of exchange, etc.); and V_1, V_2, V_3, the respective velocities of circulation, that is the ratio between the values exchanged with those instruments of circulation and the average amounts of them[7]; we may write:

$$\sum p_i q_i + M_4 = M_1V_1 + M_2V_2 + M_3V_3, \tag{11}$$

where the sum of the first side of the equation extends to the aggregate values of the goods exchanged whatever might have been the form of payment and M_4 is the amount of bills paid off.

By putting t for time and letting 0 denote a given base year, the price and trade indices may be defined

$$P = \frac{\sum p_i^{(t)} q_i^{(t)}}{\sum p_i^{(0)} q_i^{(t)}} \qquad T = \sum p_i^{(0)} q_i^{(t)}$$

[7] It is almost needless to add that if a certain exchange gives rise to transfers of several means of payment, it will be possible to separate those parts of it corresponding to transfers either of one means or of the other.

and we may write:[8]

$$P = \frac{M_1V_1 + M_2V_2 + M_3V_3 - M_4}{T} \tag{12}$$

It is clear that the difference $M_3V_3 - M_4$ depends on the changing degree of confidence and of indebtedness as well as on habit. In Italy, for instance, most bills fall due at the end of June and December; it explains why P may change, as Keynes rightly noticed, even if the other five variables of the second side of the equation remain unchanged. It also follows that in monthly or annual periods greatly disturbed by debts, insolvencies, and psychological elements, a complete statistical valuation of (12) implies the knowledge of the amount of instruments of credit on maturity, both circulating and paid off, and of the velocity of circulation of the former.

On the other side P may be considered as the weighted arithmetic average of the ratios:

$$P_1 = \frac{M_1V_1}{T_1} \tag{13}$$

$$P_2 = \frac{M_2V_2}{T_2} \tag{14}$$

$$P_3 = \frac{M_3V_3 - M_4}{T_3} \tag{15}$$

assumed with weights proportional to the denominators, which are respectively the indices of the quantities exchanged with money (T_1), with substitutes (T_2), and with bills payable on maturity in circulation (T_3). It is $T_1+T_2+T_3=T$.

The first ratio can immediately be transformed into the "Cambridge equation," putting:

$$H_1 = \frac{T_1}{V_1}. \tag{16}$$

Indeed from (13) we draw:

$$P_1H_1 = M_1 \tag{17}$$

where H_1 is the amount of goods represented by money (either in the hands of private individuals or of banks). This amount is represented

[8] I. Fisher, *The Purchasing Power of Money*, Appendix to Chapter V (Effect of Time Credit on Equation of Exchange). In this appendix, the introduction of the promises of payment M_3 and M_4 in the equation of exchange is attributed to A. Piatt Andrew. The factor V_3 was added by me.

in a unit of measure formed by that qualitative and quantitative stock of goods exchanged for money, on whose base P_1 was calculated.

The formula (16) shows that the velocity of circulation of money changes in inverse ratio to the cash resources (H_1) of the community.

By calling R the amount of goods possessed by the community, expressed in the unit of measure of H_1 and K the proportion of R held in cash, c the proportion of money held by single individuals and h the proportion of money held by banks as a reserve of bank deposits, we may write (17) in the form given by Pigou:

$$P_1KR[c + h(1 - c)] = M_1, \tag{18}$$

where P_1 is still the index of prices of the "equation of exchange," when the latter is limited to the transactions with money.

We may therefore infer that the "Cambridge equation" is a part of the "equation of exchange" expressed in a different form.[9] Let us add that the "equation of exchange," given by (12), may be expressed in the following alternative form.

If we put:

$$H_1 = \frac{T_1}{V_1}, \qquad H_2 = \frac{T_2}{V_2}, \qquad H_3 = \frac{T_3}{V_3}$$

from (13), (14), and (15) we get:

$$P_1H_1 = M_1$$
$$P_2H_2 = M_2$$
$$P_3H_3 = M_3 - M_4/V_3$$

and summing up both sides we obtain

$$P_1H_1 + P_2H_2 + P_3H_3 = M_1 + M_2 + M_3 - M_4/V_3 \tag{19}$$

and also:

$$P'(K_1R + K_2R + K_3R) = M_1 + M_2 + M_3 - M_4/V_3 \tag{20}$$

where:

$$K_1R = H_1$$
$$K_2R = H_2$$
$$K_3R = H_3.$$

Now R is expressed in a unit of measure formed by that greater qualitative and quantitative stock of goods on the base of which P' is

[9] The formula (16) was developed by Dionisio Tenderini in his graduation thesis at the School of Commerce of Venice. By putting again $H_2 = T_2/V_2$, he could pass from one formula to another, limited to money and substitutes. By considering instruments of credit payable on maturity, I infer (19).

calculated, and K_1, K_2, K_3 are the rates of goods of the community held respectively in the form of money, substitutes, and instruments of credit payable on maturity.

Let us point out that P' is the weighted arithmetical mean of P_1, P_2, P_3 taken with weights respectively equal, no longer at T_1, T_2, T_3 (as in the "equation of exchanges"), but at $H_1=T_1/V_1$, $H_2=T_2/V_2$, $H_3=T_3/V_3$.

In Pigou's notations we may put:

$$K = K_1 + K_2$$

and write:

$$K_1R = KR[c + h(1 - c)]$$
$$K_2R = KR[(1 - h)(1 - c)].$$

Likewise in K_3R we could distinguish the debts payable on maturity of private individuals from those of banks.

Let us call (20) the "generalized Cambridge formula." Of course we do not claim that by the mere formality of using (20) we would obtain much better results than by using (18). A real progress would be made only by going more deeply into the interrelations of the individual terms of which the equations consists, and into the interrelations of other factors, with which they are connected.

Let us consider now the two formulas of Keynes. They do not differ very much from the preceding ones, even if for the first of them we take into consideration Hansen's remark and Lundberg's correction.[10]

Keynes takes for granted that in a certain community and in a unit of time the level of prices of the *consumers' goods*, multiplied by the volume of their production ($\overline{P}\,\overline{R}$), that is, the expenditure for consumers' goods, is equal to the total monetary income or to the earnings of the public (E) less saving (S):

$$\overline{P}\,\overline{R} = E - S. \tag{21}$$

By resolving E into two parts, considering whether the income was got through consumers' goods or through investment goods:

$$E = E_r + I'$$

we get the first formula:

$$\overline{P} = \frac{E_r}{\overline{R}} + \frac{I' - S}{\overline{R}}. \tag{22}$$

[10] This correction was suggested by A. H. Hansen and Herbert Tout, in their paper, "Investment and Saving in Business Cycle Theory," in ECONOMETRICA, April, 1933.

The level of prices of consumers' goods therefore consists of two terms: the first term representing the rate of earning, per unit of production, of consumers' goods, which may be considered equal to the unit cost of their production; the second term is a fraction where the numerator is the difference between expenditure (or total cost) of the investment goods and the amount of saving, and the denominator is the volume of production of consumers' goods.

It is obvious that $\overline{P}$ is a partial index as compared with P or P' of the formulas (12) and (20), but it is no longer distinguished according to the means employed in the payments, but by considering the quality of acquired goods, that is, consumers' goods. The formula (21) expresses that the expenditure which such consumers' goods brings on the market is equal to the monetary income appointed to their purchase; in other words the monetary income in question is not distinguished according to the means of payment employed, but rather according to the purpose and the source of income. Besides this, Keynes takes for granted that in the time and place considered, the level of prices of the *whole production* multiplied by the volume of whole production (ΠO), i.e., the expenditure for every kind of goods, is equal to the total monetary income or to the earnings of the public (E), added to the windfall profits (Q) of the entrepreneurs, besides their normal remuneration:

$$\Pi O = E + Q. \tag{23}$$

As saving (S) and profits (Q) are not considered in the expense for consumers' goods $\overline{PR}$, we must put:

$$Q + S = I,$$

where I is the total value of the new investment goods; therefore from (23) we draw:

$$\Pi = \frac{E}{O} + \frac{I - S}{O} \tag{24}$$

which is the second formula of Keynes.

The level of prices of all the goods produced consists therefore of two terms: the first term represents the rate of earning, per unit of production, of all the goods produced, which may be considered equal to the unitary cost of their production; the second term is the unit profit (equal to the difference between the total value of the investment goods and the amount of saving) divided by the volume of total production.

Also Π is a partial index as compared with P and P', because it concerns the volume of *production*, which is quite different from the vol-

ume of the *exchanges* or of the goods possessed by the public, so that we can repeat for it what we said concerning $\overline{P}$.

From (24) we deduce that, when the level of prices equals the rate of earning per unit of production, we have $I=S$, and the corresponding mean rate of interest is the mean rate of equilibrium or natural rate (according to Wicksell); while when investments are greater or smaller than savings we have a superior or inferior level of prices of production.

Having taken this for granted, it was easy for Mr. Keynes to explain how the fluctuations in prices of commodities and business cycles are due to a lack of balance in the values of investments respecting the amount of savings and how banking policy may control them by managing the rate of discount and bank deposits in such a way as to modify the distribution of income between consumption and savings and between investments and bank deposits on the part of the public, and the distribution of production between consumers' goods and investments on the part of entrepreneurs.

Keynes properly calls "credit cycles" those due to the fluctuations of the second term (24) of which he gives examples of some determinant factors, and "income inflation" or "income deflation" the increase or fall of the first term.

Now—without considering the fact that the second equation of Keynes is bound to a particular definition of profit—those who remember the theory of economic equilibrium will be easily persuaded that even on the simple statical hypothesis, the equality between savings and investments is *not* sufficient to determine the mean rate of interest, the latter being connected with a great number of other factors, which in their turn have an influence on savings and investments.

Following the order of ideas which led Keynes to develop the formula (24), we must remark that the fluctuations of the two terms of the second side are closely connected with each other: those banking transaction that aim to modify $I-S$, necessarily disturb E/O because the movements of discount and bank deposit have an influence also on the income of the public (E) and on the volume of production O, and such influences take place through a sequence of repercussions that (24) conceals from us and about which we cannot make sufficient conjectures.

There will be disputes, contentions, new points of view, even clever and perspicuous ones as those of Hayek,[11] Robertson, Robbins, and other students, *but we shall always be at about the same point as before.*

It can never be sufficiently repeated that, when we are discussing

[11] A greatly elaborated critical analysis of Hayek's thesis was made by Hansen and Tout in the above-mentioned article. A remarkable study on "Consumption and the Trade Cycle" was published by L. Robbins in *Economica*, November 1932.

questions involving quantitative economic variables, it is necessary first to map out their connections formally with the help of reasoning, and then by means of empirical research to verify the actual "routine of change" of the phenomena, and *express these empirical facts in our equations.*

While remaining in the field of business cycles and getting profit from Keynes's analysis, L. Amoroso in a recent course of lectures[12] proposes a system of equations, some of which do not seem to us difficult to verify empirically or to correct according to the empirical results obtained.

Amoroso's system has the merit of showing how it is possible to connect the equations of partial equilibrium with the empirical equations, and how the properties of the various equations may be examined by experience.

First of all he identifies the "industrial system" by the five variables:

$\overline{R}$: volume of production of consumers' goods.
$\overline{\overline{R}}$: volume of production of investment goods.
a) E: total monetary income (that is the total cost of production, profits or losses of entrepreneurs excluded).
Q_1: profits (or losses) of entrepreneurs on consumers' goods.
Q_2: profits (or losses) of entrepreneurs on investment goods.

and the "commercial system" by the following five variables:

$\overline{P}$: index of prices of consumers' goods.
$\overline{\overline{P}}$: index of prices of investment goods.
b) K: amount of income spent.
S: amount of income saved.
D: amount of saving invested (proportion $S-D$ is hoarded).

Then he writes five equations, showing how the movement of the commercial system has an influence on the industrial system, and five other equations, showing how the second system has an influence on the first.

Then he shows *circuits of disequilibrium* and *circuits of compensation* among these ten variables and explains the work of the banking system in checking the circuits of disequilibrium and in helping towards compensation.

However, following Hansen's correction, we think it better to reduce the variables to eight, by putting in place of Q_1 and Q_2 the total profits Q, and by neglecting K, which is nothing else but $\overline{P}\overline{R}$ and gives rise to a trivial identity. The following analysis, therefore, proceeds on

[12] Luigi Amoroso, *'La Dinamica dei Prezzi."* Roma, Gruppo Universitario Fascista, 1933.

lines different from those indicated by Amoroso. I consider the following 8 equations:

$$\text{I.}\quad \overline{R}(t+\omega_1) = a_1 + m_1 Q'(t) + n_1[E'(t) - S'(t)]$$

$$\text{II.}\quad \overline{\overline{R}}(t+\omega_2) = a_2 + m_2 Q'(t) + n_2 D'(t)$$

$$\text{III.}\quad E(t) = \overline{P}\left(t - \frac{\omega_1}{2}\right)\overline{R}(t) + \overline{\overline{P}}\left(t - \frac{\omega_2}{2}\right)\overline{\overline{R}}(t)$$

$$\text{IV.}\quad Q(t) = \overline{P}(t)\overline{R}(t) + \overline{\overline{P}}(t)\overline{\overline{R}}(t) - E(t)$$

$$\text{V.}\quad \overline{P}(t) = b_1 - r_1\overline{R}'(t+\omega_1) + s_1[E'(t) - S'(t)]$$

$$\text{VI.}\quad \overline{\overline{P}}(t) = b_2 - r_2\overline{\overline{R}}'(t+\omega_2) + s_2 D'(t)$$

$$\text{VII.}\quad S(t) = c_1 + h_1 E'(t) - k_1\overline{P}'(t)$$

$$\text{VIII.}\quad D(t) = c_2 + h_2 E'(t) + k_2\overline{\overline{P}}'(t)$$

Equation (I) shows that the volume of production of consumers' goods represents with a certain delay the movement of profits and of incomes and savings. The volume of production shows a tendency to expand when either profits or excess of income over saving rise with a quicker rhythm, and it shows a tendency to contract in the opposite cases. We have assumed a simple linear equation in the variables $Q'(t)$, $E'(t)$, $S'(t)$; these variables are the derivatives, with regard to time, of $Q(t)$, $E(t)$, and $S(t)$. Equation (I) is the simplest equation that expresses the relation mentioned and it is convenient to keep it until its true form is revealed by empirical investigations.

Equation (II) expresses that the volume of production of investment goods depends in a similar way on profits and the invested savings.

Equation (III) expresses that the total monetary income depends on the volume of the productions $\overline{R}$ and $\overline{\overline{R}}$ both considered at the same time t, and on the corresponding indices of prices $\overline{P}$ and $\overline{\overline{P}}$, at the points of time $t-\frac{\omega_1}{2}$ and $t-\frac{\omega_2}{2}$, ω_1 being the average cycle of the production of consumers' goods and ω_2 that of investment goods.

Equation (IV) shows the profits, as being drawn from (23), resolving ΠO into the two parts corresponding to consumers' and investment goods.

Equation (V) shows that the market discounts in the present prices the future fluctuations of consumers' goods and it immediately shows the movement of the increase of incomes over savings.

Equation (VI) shows that the index of prices of investment goods depends in a similar way on the variations of investment goods and on the amount of the invested savings.

Equation (VII) shows that the amount of savings depends on the

variations of the total monetary income and on the index of prices of consumers' goods.

Finally, equation (VIII) shows the analogous connections between the amount of invested savings, the total monetary income, and the index of prices of investment goods.

The decisions of entrepreneurs are represented by (I) and (II); those of savers by (VII) and (VIII); the reactions of the market by (V) and (VI). Finally (III) and (IV) are two identities.

Equations (I) and (II) connect, via (IV), the price variations with the future volume of production, therefore of trade; but our equation (7) concerns two special groups of prices.

Assuming that the eight equations are significant, at least with regard to the nature of the variables appearing in each of them and to the general character of the relations, we may say, for instance, that $\overline{P}$ has:

1) a *primary direct influence* on E and Q (equations (III) and (IV)) and a *primary inverse one* on S (equation (VII));

2) a *secondary direct or inverse influence* through E, on $\overline{R}$, Q, $\overline{P}$, S, D; through Q, on $\overline{R}$, $\overline{\overline{R}}$; through S, on $\overline{R}$, $\overline{P}$;

3) a *tertiary direct or inverse influence*

through $E\overline{R}$, on E, Q, $\overline{P}$
through EQ, on $\overline{R}$, $\overline{\overline{R}}$
through $E\overline{P}$, on E, Q, S
through ES, on $\overline{R}$, $\overline{P}$
through ED, on $\overline{R}$, $\overline{P}$
through $Q\overline{R}$, on E, Q, $\overline{P}$
through $Q\overline{\overline{R}}$, on E, Q, $\overline{P}$
through $S\overline{R}$, on E, Q, $\overline{P}$
through $S\overline{P}$, on E, Q, S

and so on through the quaternary *direct or inverse influence*, etc.

We consider, for instance, the secondary influence exerted by $\overline{P}$, through E, on $\overline{P}$: we have the circuit $\overline{P}E\overline{P}$, expressing that the movement of $\overline{P}$ has an influence on that of E (equation (III)) and the latter has an influence on $\overline{P}$ (equation (V)), but it is a *circuit of disequilibrium*, because the movement of $\overline{P}$ in a certain direction tends to develop a new movement *in the same direction*.

On the contrary, if we consider the tertiary influence exerted by $\overline{P}$, through $E\overline{R}$, on $\overline{P}$, we find that it forms a *circuit of compensation:* $\overline{P}E\overline{R}\overline{P}$, because according to (III) and (I) the two influences are *direct;* while, according to (V), the third influence is *inverse*, so that in such a circuit the movement of $\overline{P}$ in a certain direction tends to develop, as a

reaction, a movement of $\overline{P}$ in a contrary direction. The same applies to the circuit $\overline{P}ES\overline{P}$.

The circuits of disequilibrium are the expression of the forces of disequilibrium of the system, that is, of those factors showing a tendency to bring the system further and further away from the initial position.

Banks show a tendency to check the action of the forces of disequilibrium and to help towards compensation, above all by managing money, bank deposits, debts payable on maturity, and the rate of interest. For instance, by managing the rate of interest, the prices of some investments included in $\overline{\overline{P}}$ are varied: they are founded on current rates of interest and therefore (by (III) and (IV)) there is a tendency to modify E and Q in the same direction and (by (VIII)) also D in the same direction. So that there are raised many circuits, which other banking practices will be able to accentuate or to reduce, in connection with the new tendencies of savers and entrepreneurs.

In concluding I would like to say that in my opinion it is only by basing the analysis on *systems of dynamic equations* such as (I)–(VIII), or similar equations, that a really scientific theory of business cycles can be developed.

University of Bologna

DEMAND STUDIES DURING TIMES OF RAPID ECONOMIC CHANGE

By E. J. Working

Paper read before the Econometric Society, Philadelphia, December 28, 1933

The rapidly changing economic conditions of the past four years have confronted those of us who study demand through the use of statistics with some very serious difficulties. Nevertheless, they have also presented us with new opportunities. They have helped to shed new light on the nature of the demand for some commodities and to broaden the scope of statistical studies of demand. As a result of the broadening of their scope, I believe that these demand studies are now in position to contribute to the better understanding of two of the most important problems with which economists deal. These problems are: What are the causes of fluctuations of business activity? What are the causes of changes in the price level?

Now, lest I be misunderstood, let me remind you that I have not said that demand studies will provide a complete answer to these questions. I have only said that they can contribute to the better understanding of these problems.

Statistical studies of demand may be classified from various standpoints. I am not here concerned with the budget studies, such as have been pioneered by Pigou, nor with studies using the Leontief method. I shall discuss only the more usual sort of study which deals with time series data and which uses correlation methods in the analysis of these data. Such studies may be classified into two general types which for convenience I shall term the "old" and the "new" types of demand studies. It should be understood at the outset, however, that there is no hard and fast line of distinction to be drawn between them. One type merges into the other and, furthermore, the studies which I class as belonging to the new type are not altogether confined to those which have been made in recent years.

The older type of study had as its principal purpose the determination of the demand curve of a commodity—or, in some cases, the more limited purpose of the determination of the elasticity of demand. From its standpoint, changes in the conditions of demand were to be regarded as the result of "disturbing factors," and hence the effect of these factors should be removed from the data. Various devices were used in pursuance of this end, including link relatives, first differences, and trend ratios. Even multiple correlation was used in order to *elim-*

inate the effect of disturbing factors.[1] On the whole, however, this older type of study was likely to proceed on the assumption that changes in demand were largely of a gradual character, due to changes in habits, customs, and the growth of population. Under such conditions, of course, the use of methods to remove the effect of trends would be relatively successful.

The newer type of demand study has for its purpose not only the approximation of a demand curve. It also attempts to analyze the changes in demand. Many of the studies which have been termed "price analyses" are to be classified as belonging to the new type of demand studies. Often, however, the term price analysis has been applied to studies which differed little from the older type, and "explained" a large part of price variations in terms of trends or by the use of time as an independent variable. In still other cases the attempt was to arrive at an empirical forecasting formula and scant attention was given to the analysis of causal relations.

Most of the statistical studies of demand have been made since the World War, but it has been common practice to omit the war years and immediate post-war years in these analyses.[2] For example, a given study may cover the period 1900 to 1916 and 1921 to 1928. The obvious reason for omitting the war years in the case of many studies is that when those years were included results were not so "satisfactory." Either the investigator did not succeed in explaining price changes during the war as readily or as completely as the pre-war and post-war changes, or he did not consider the regression of price on quantity to be as reliable an indication of the demand curve when those years were included. Still other studies have been confined to post-war years only, and these have seldom used years prior to 1921.

Now, I do not wish anyone to think that I look with disfavor upon

[1] For example, Schultz wrote ". . . The method of multiple correlation is, as we have just seen, the best method for dealing with the problem of disturbing factors. . . ." and "*All of these statistical devices are to be valued according to their efficiency in enabling us to lay bare the true relationship between the phenomena* under consideration. An *Ideal* method would eliminate entirely all the disturbing factors." Schultz, Henry, *Statistical Laws of Demand and Supply*, University of Chicago Press (1928), pp. 31 and 33.

[2] A very large number of "price analyses" or other types of studies involving attempts to approximate the demand function have been made by various investigators in the past ten years. Though some results of a number of these studies have been published, the great majority have not and presumably never will be published. The lack of such published results, of course, places one who wishes to generalize concerning their nature in a rather difficult position, and makes his opinion subject to considerable error. I hope that anyone who disagrees with my opinion as to the general nature of these studies will not hesitate to let me know of his disagreement.

the omission of the war and immediate post-war years. Indeed, in the case of many commodities, the forces in operation during a part of this period were so unusual that I am inclined to look with especial skepticism upon any statistical study which presumes to explain the resulting prices. I wish, however, to point out that while there are numerous exceptions, studies made prior to 1930 commonly did not include any period of prolonged and drastic business recession and covered periods of time which were relatively free from cyclical changes in the conditions of demand.

In the past four years it has become evident that many analyses of prices made prior to 1930 were faulty. In many cases the difficulty was that the only changes in demand of which specific account had been taken were trends. This was, in turn, due to the fact that shifts in demand were small during the period covered by the analyses; or that such shifts were closely correlated with changes in the price level or with other factors, which, although included in the study, were not intended to explain shifts of demand. Thorne and Bean have shown, for example, that what was formerly treated as merely a long-time upward trend in the demand for beef and pork was primarily a cyclical increase in demand—the result of increasing business activity and pay rolls.[3] They further showed that a very different approximation of the demand curve for beef was obtained as a result of their better analysis.

Prior to 1930, the newer type of study was commonly handicapped by being limited to periods of time during which there were no very great changes in demand which were not readily "explainable" on the basis of trend. If the basic data of a study do not cover a period of time during which there have been marked cyclical changes in the level of demand, it is manifestly not likely that an analysis of them can be made to show what factors would be responsible for such shifts of demand in some other period of time. Correlation studies cannot be expected to give any clue to the nature of the causal relationship which exists between price and some other variable which has not varied significantly in the period under observation.

There were, of course, important shifts of demand which occurred during the war and the two following years which might well have been analyzed, but as a rule they were not analyzed. Nevertheless, there were many aspects of the war experience which were so wholly unusual as to make that period unsuitable as an indication of what might be expected to cause changes in demand during a more normal period. Price-fixing, publicity drives designed to modify consumption, and the way in which wartime needs impinged upon individual commodities, con-

[3] Thorne, G. B., and Bean, L. H., *Journal of the American Statistical Association*, XXVII (March 1932), 61–67.

tributed to obscure the relationship between changes in demand and the forces which might be expected to cause such changes during normal times. As one goes back in search of still earlier periods of time for which analyses of cyclical changes in demand may be made, he will find no depression of correspondingly great severity since the turn of the century. Furthermore, as he attempts to go back further and further in order to study the causes of changes in demand, he finds himself dealing with a period where the statistics he needs become more and more scarce and their reliability more and more doubtful. Then, too, in periods approaching a half century in length, it is naturally to be expected that various institutional changes will materially alter the interrelations between concurrent factors.

The very abrupt decline in the level of demand for most products during the past four years has consequently served not only to emphasize the need for studying shifts in demand, it has also provided a much better opportunity for studying these shifts.

There is still another way in which the changes of recent years have aided studies in demand. We have many commodities the money price of which is substantially fixed. If the price is fixed, how can one, from statistical data, obtain any clue as to the extent to which changes in the price would have affected the quantity demanded? If there is no change in the general price level, we cannot—at least not by the usual time-series methods. However, when we have a period for study during which there is a great change in the price level, those commodities for which the money price is fixed undergo a great change in "real" prices. With such a change in real prices the way is opened to arrive at some statistical evidence concerning the effect of those changes upon the quantity of the commodity demanded. An example of such studies is to be found in the work of Bean and Whitman on pig iron prices. Similarly, I was able last year to present some statistical evidence as to the nature of the demand for the entire group of industrial products as a whole. These are studies which, faulty as they may be, could scarcely have been made had it not been for the great decline of the price level during the past four years.

As an illustration of a type of analysis which has been successfully used in studying the cyclical shifts in demand, I have here a chart depicting a study made by Mr. G. B. Thorne and Mr. Preston Richards of the Bureau of Agricultural Economics (Figure 1). This study covers the period 1921 to 1931, but I may add that it satisfactorily explains prices for the years 1932 and 1933 if we merely extend the regression of price deviations on some index of consumer incomes. The retail price of pork is here shown to be very highly correlated with the consumption of pork and an index of consumer incomes. You will note, perhaps,

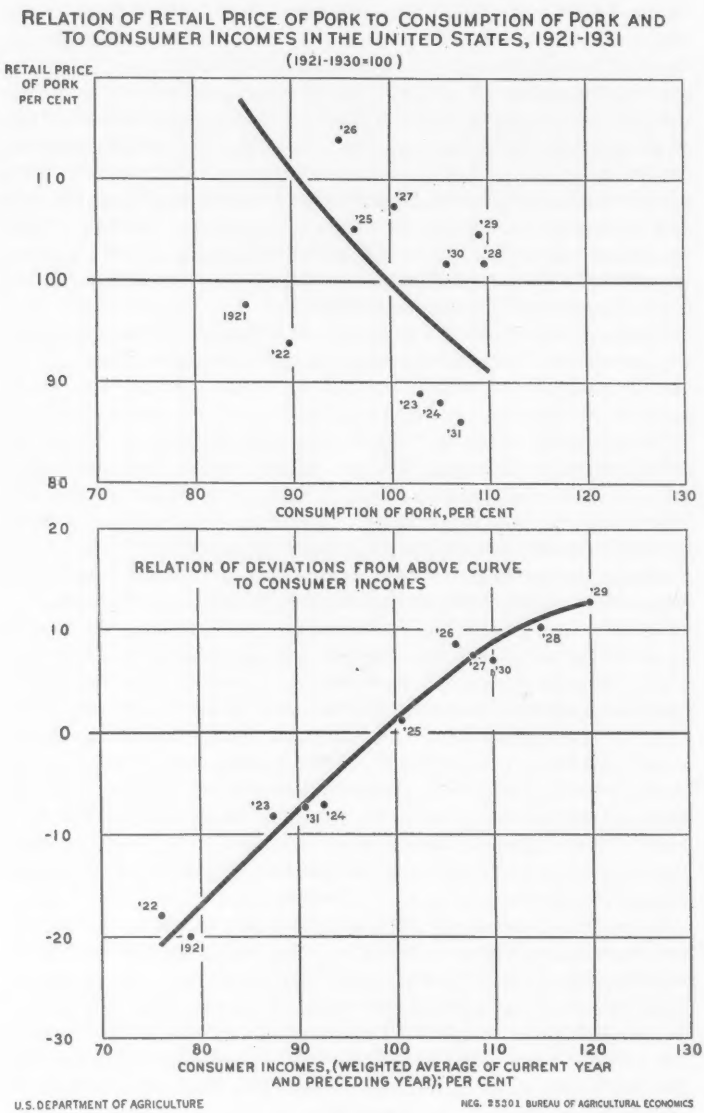
RELATION OF RETAIL PRICE OF PORK TO CONSUMPTION OF PORK AND TO CONSUMER INCOMES IN THE UNITED STATES, 1921-1931
(1921-1930=100)
RETAIL PRICE OF PORK PER CENT
110
100
90
80
70
80
90
100
110
120
130
CONSUMPTION OF PORK, PER CENT
'26
'27
'25
'29
'30
'28
1921
'22
'23
'24
'31
RELATION OF DEVIATIONS FROM ABOVE CURVE TO CONSUMER INCOMES
20
10
0
-10
-20
-30
70
80
90
100
110
120
130
CONSUMER INCOMES, (WEIGHTED AVERAGE OF CURRENT YEAR AND PRECEDING YEAR); PER CENT
U.S. DEPARTMENT OF AGRICULTURE
NEG. 25201 BUREAU OF AGRICULTURAL ECONOMICS

FIGURE 1

that the data of this chart are in terms of per cent. These, however, are percentages of the 1921–1930 averages, not of trends, so that precisely the same results would be obtained by using actual prices and actual pounds of pork consumed. The prices used were money prices—not prices adjusted for changes in the price level. From this analysis it appears that "money demand" for pork apparently has fluctuated very closely with changes in the index of consumer income.

The term "money demand," I use in contrast to "real demand." By the former, I mean the relation between quantities demanded and money prices; by the latter the relation between quantities demanded and "real" prices—that is prices which have been adjusted for changes in the price level. The distinction between these two is highly important from the standpoint of the person interpreting the analysis as well as from the standpoint of the person making it. In times of changing price levels, it means nothing to say that demand has decreased or that demand has increased, unless it is clear whether "money demand" or "real demand" is meant. Much that has been written concerning changes in demand of the past few years is subject to this confusion. The greatest danger involved in such confusion is that an argument may without notice shift from the evidence of a change in "money demand" to the effect of a change in "real demand."

From the standpoint of price analysis, however, there is another very important aspect of this question. If "money prices" are used, great care should be exercised in considering whether the demand factor (or factors) used may be expected satisfactorily to account for changes in "money demand" rather than for changes in "real demand." Among the factors which have been used to account for shifts in demand are indexes of business activity, of industrial production, of employment, of payrolls, of commodity prices, and of consumer incomes. Save in the amplitude of their fluctuations, the course of all these indexes from 1921 to 1932 was very similar. Generally speaking, one can be substituted for another in a correlation with relatively little difference in the correlation coefficient obtained. From 1933 on, however, there are likely to be very important differences. When "money prices" are used, it is highly essential that shifts in demand be related to a value index such as consumer incomes, or payrolls, whereas a volume index such as industrial production should be used in any attempt to explain changes in real demand.

It is not always possible to explain cyclical shifts in demand on the basis of changes in industrial production, consumer income, or similar indexes of United States conditions. Indexes which are closely associated with the conditions of demand for domestic commodities are likely to be inadequate for use in analyzing shifts of demand for inter-

national commodities. A case in point is wheat. I have here a chart (Figure 2) showing certain price and supply relationships for that commodity. The upper section shows the relation between the price of wheat in Great Britain, adjusted for the British price level, and the estimated supply of wheat for the world excluding Russia and China. You will note that this section is somewhat similar to the upper section of the pork chart in that there tend to be two distinct groupings of the price-quantity observations. The data for the crop years 1921–22 to 1923–24 and for 1930–31 to 1932–33 group themselves fairly closely about what might be termed a "low demand curve," while the data for the other years are fairly close to the "high demand curve." In view of the similarity, it may appear that these data might be treated in the same manner as were those of the pork price study.

As a matter of fact, a procedure similar to that used in the pork price study does yield a high correlation. However, the careful price analyst will not, upon obtaining such a correlation, rush forth with the explanation that the price of wheat at Liverpool is determined by the supply of wheat in the world (outside Russia and China) and industrial production in the United States. Instead he will ask himself, "Why do I get so high a correlation?" A partial answer to that question is, of course, to be found in the fact that the fluctuations in business activity of the more important wheat-consuming countries of the world have been quite closely correlated in the past fifteen years.

Before you put too much weight upon that explanation, however, let me call to your attention the fact that there are a number of important wheat consuming countries in which wheat prices have not followed a course parallel to that of British wheat prices in the past four years. In some of these, "money" prices are now as high as, and "real" prices are higher than, in 1926–27. In an attempt to take some account of the widely different prices in different countries, I have had computed a "world" average "real" price of wheat. This was done by first adjusting the money price of wheat in each of seven important wheat consuming countries for changes in the price level of those countries, and converting these adjusted or "real" prices to terms of gold cents per bushel at the par of exchange. A weighted average of these seven series was then computed—the weighting being in proportion to the wheat consumption of the regions for which the original price series were deemed to be representative. The resulting prices as plotted against the wheat supply for the world (excluding Russia and China) are shown in the lower half of Figure 2. You will note the years 1930 to 1932 here appear close to the higher curve.

In passing, let me point out that the data depicted on this chart are all unadjusted except for the dividing of prices by "all-commodity"

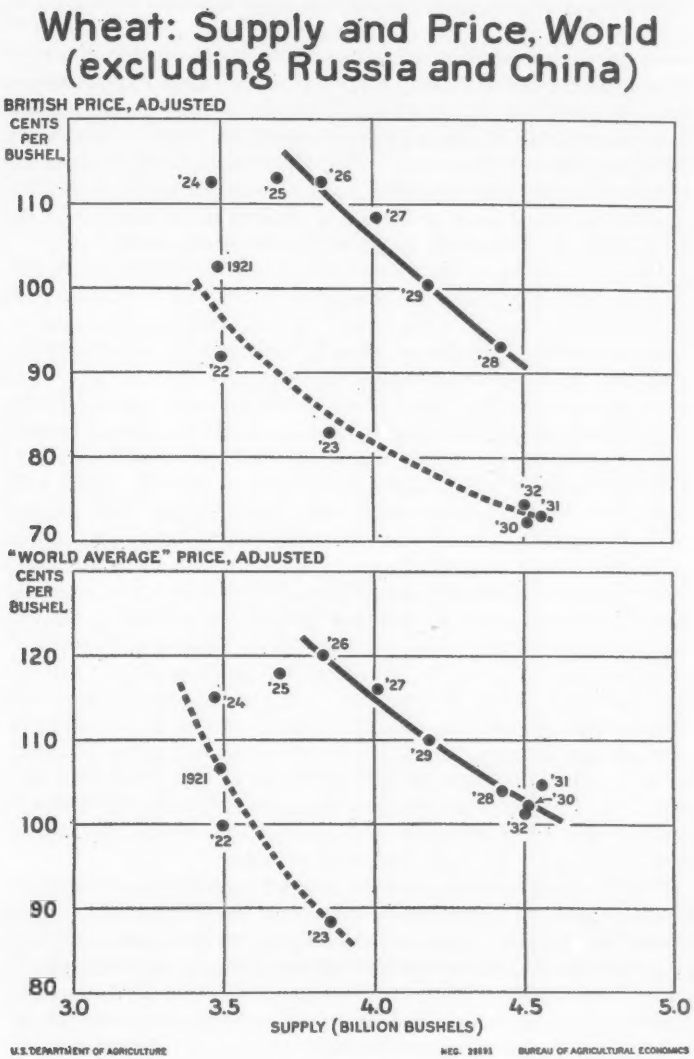
Wheat: Supply and Price, World
(excluding Russia and China)
BRITISH PRICE, ADJUSTED
CENTS PER BUSHEL
110
100
90
80
70
'24
'25
'26
'27
1921
'29
'28
'22
'23
'32
'31
'30
"WORLD AVERAGE" PRICE, ADJUSTED
CENTS PER BUSHEL
120
110
100
90
80
'26
'25
'24
'27
'29
1921
'31
'28
'30
'32
'22
'23
3.0
3.5
4.0
4.5
5.0
SUPPLY (BILLION BUSHELS)
U.S. DEPARTMENT OF AGRICULTURE
NEG. 28695
BUREAU OF AGRICULTURAL ECONOMICS

FIGURE 2

price indexes of the various countries in order to obtain a "real price" in place of a "money price." The chart was made in this manner in order to present the raw data rather than data in which trends have already been taken out. I may say, however, that there appears to be a gradual upward trend in the demand for wheat, and if an allowance is made for this, the dots would fall more nearly on the two lines shown. The lines, however, then have a somewhat steeper slope, indicating a greater inelasticity of demand than that suggested by the chart. The demand for wheat for consumption as opposed to the demand for the total supply is of course indicated to be even more inelastic.

I have not the time, nor is this the appropriate place, to go into the many details of evidence concerning the demand for wheat. Furthermore, I believe that what I have already said is sufficient to make my next point clear. It is that we are still faced with some very difficult problems when we attempt to analyze changes in demand. One of these is due to the fact that demand fluctuations are in part, at least, of a cyclical nature. It is largely because of this fact that it seemed necessary to point out earlier in this paper that care should be used in selecting an index which might be expected to move in accord with changes in the demand of a commodity. Indexes of industrial production, employment, payrolls, and consumer income, as well as various price indexes, are of a cylical nature, and they all have moved fairly closely together in the past fifteen years. The fact that a given one of these indexes seems to be closely related to fluctuations of demand during the past fifteen years is of itself of little value in indicating how that index will be related to the cyclical shifts in demand for a particular commodity during future years. The close relation between two series of data through one complete cycle cannot be considered to be of much significance as empirical evidence. Even though the data may cover a period of a dozen years, this does not mean that we have that many independent observations. There may, indeed, be question whether data which cover one cycle should be considered the equivalent of as many as two independent observations.

This brings us again to the point which has often been stressed before; the need for the analysis of demand in terms of causation. This is especially important where we are studying changes in demand which are primarily of a cyclical nature, and for which we have only one or two cycles of evidence. Closest attention must in such cases be given to *a priori* reasoning concerning the causal relations which may be involved, and to the many scattered bits of inductive evidence which may bear upon the problem. Even where data are not of a cyclical nature, statistical methods alone are incapable of yielding definite evidence of causation. It is only when we can combine statistical evidence

in a closely knit reasoning process that we can hope to arrive at causal, and hence permanent, relationships between factors. This, I take it, should be *a* prime, if not *the* prime interest of members of the Econometric Society.

The question of whether we should use money prices or "real" prices in measuring demand has been a matter of general concern among price analysts for some years. In recent months another question has come to the fore. Should the prices used be currency prices or gold prices?

In order to make the question more concrete, let us consider Figure 3 which shows the deflated price of cotton (the price adjusted for price level changes) in relation to the supply of American cotton. Let us

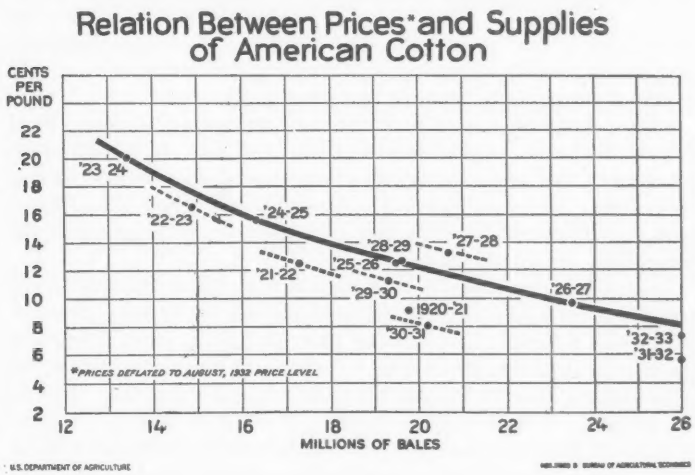

FIGURE 3

assume for the sake of simplicity that, insofar as past analysis indicates, demand conditions for the 1933–34 season are similar to those of 1926–27. Under such conditions, and with a supply of about 24,800,000 bales, what money price will result? The chart would indicate about 9 cents per pound as the deflated price, but these prices on the chart all refer to years when a dollar was worth approximately as much as one-twentieth ounce of gold. With the dollar now worth only about one thirty-fourth as much as an ounce of gold, will the money price be increased accordingly?

The procedure which should be followed is not the same for all com-

modities. In the case of such commodities as beef and pork, merely taking account of changes in the "all commodity" price level should give the approximate result. On the other hand, in the case of a commodity such as wheat, it would be better to estimate what the price level would have been if we had remained on the gold standard, adjust for the difference between such estimate and the base period price level, and then multiply the result by 1.69 or whatever is deemed to be the probable ratio of the price of gold for the year to the old price of $20.67 per ounce. In the case of cotton, I should say that one comes nearer the truth by adopting the procedure outlined for wheat. Nevertheless, so large a part of the demand for cotton is dependent upon conditions in the United States that the price would probably fall short of that which would be arrived at by this method. The best procedure would be, of course, to study separately the demand for the commodity in the United States and the demand in foreign countries. The results may then be synthesized in the light of the divergent movements of foreign exchange rates and price levels.

I have already indicated that with the development of what I have termed the "new" type of demand study, and with the improved results which that type of study has been able to show within the last four years, there has been a broadening of the scope of statistical studies of demand. We have also advanced to new frontiers in the application of demand studies.

In any consideration of business cycle phenomena, changes in demand are of prime importance. For example, one of the moot questions with which students of the business cycle are dealing today is whether a decline in industrial production is caused by a decline in demand for industrial products, or whether the reverse is true—that the decline in the demand which takes place during a business depression is the result, rather than the cause, of smaller industrial production. Thus, to many people the real need at a time of depression is to raise wages in order that, once wages are raised, consumers will have more purchasing power, and then production will be increased. If, however, the direction of causation is the reverse, so that production needs to be increased before real purchasing power can be raised, a very different line of procedure is indicated from that based on the other theory. My own opinion is that the causal forces run in both directions, but that basically demand and real income depend on production. In any event, it is to this sort of question that I believe studies of demand can be directed with profit in order that we may have a fuller understanding of the forces with which we must deal if we are to control business fluctuations.

Another of the new frontiers which we have reached in the applica-

tion of demand studies relates to the factors which affect the level of commodity prices generally. If we can do a creditable job of analyzing the factors which affect the price of a considerable number of important commodities, we are in position to put our knowledge concerning these individual commodities together and arrive at some conclusions concerning what factors affect the general level of prices. This is true even though our analysis of the prices of individual commodities be in terms of "*real*" or *deflated* prices.

Last year in a paper before this Society I pointed out that, because of the rigidity of the money prices of many industrial products, a decrease in the production of such products, once initiated, tends to result in a fall of the general level of prices. This is because a reduction in the production of any large group of commodities reduces the "real" demand for commodities generally. In the case of commodities for which money prices are substantially fixed by custom or monopoly, this will result in further reductions of production, but, in the case of other commodities, a fall in their "real" prices results. With "real" prices of some commodities declining while money prices of others remain stationary, there must of course be a decline in the average level of money prices.

Such are the broad outlines of what I conceive to be a most important phase of the interrelations of production and prices in an economic system where imperfect competition and other institutional factors result in partial price rigidity. Further analysis along this same line yields significant results in appraising the factors which affect price levels under conditions where there is no depreciation of a country's currency. The application of knowledge gained in the analysis of demand of individual commodities can also yield a very important contribution concerning the extent to which, and the rapidity with which, the average level of prices can be influenced by currency depreciation or devaluation.

Bureau of Agricultural Economics
Washington, D. C.

CAPITAL AND INDUSTRIAL FLUCTUATIONS

By F. A. von Hayek

A sympathetic criticism of the kind to which the views of the present author have been subjected by Messrs Hansen and Tout in a recent issue of Econometrica,[1] offers a welcome opportunity of clearing up some points upon which I have obviously not yet been sufficiently explicit. The critical comments of the two authors are mostly directed against points where real difficulties present themselves; and while I think I can answer their main objections, it is probable that I can do so more profitably by means of a further systematic development of my thesis than by wasting time on the comparatively unimportant discussion of whether these developments were already implied in my earlier statements, or whether the interpretation put upon these by Messrs Hansen and Tout can, or cannot, be justified from the admittedly sketchy and incomplete exposition in *Prices and Production*.

I

With one exception, I fully agree that their formulation of my views into the ten theses is a fair and accurate summary of my position. Even the unimportant exception is probably only a slip of the pen, and is satisfactorily cleared up in the later discussion. But, as it may have confused some readers, I should like to emphasize at the outset that I should never say, as stated in thesis number 2 (p. 133) that forced saving can ever *directly* bring about a shortening of the process of production. Forced saving means essentially a lengthening of the process of production and the crucial point is that, in my view, it is these elongations which are likely to be partly or wholly *reversed* as soon as the cause of the forced saving *disappears*.

The first major difficulty which Messrs Hansen and Tout discuss, is connected with what they call my thesis number 1, namely, that the phenomenon of depression is equivalent to a shrinkage of the structure of production. Their difficulty here seems to me to turn on the distinction between a completed and an uncompleted structure, which I have probably failed to make sufficiently clear, and which is closely connected with the distinction between the effects of mere fluctuations in the rate of saving (or, more correctly, in the rate of investment) and the peculiar instability of capital created by means of forced saving. The best way of making these distinctions clear is probably to start with a general discussion of the effects of fluctuations in the

[1] A. Hansen and H. Tout, "Annual Survey of Business Cycle Theory: Investment and Saving in Business Cycle Theory," Econometrica, I (1933), 119.

means available for investment on the structure of production in general, and on the profitableness of the early stages in particular. From this discussion it will, I think, appear that, contrary to the opinion of Messrs Hansen and Tout, it is not the mere fluctuations in the rate of investment which tend to make the earlier stages unprofitable but only, on the one hand, particularly violent fluctuations of this sort, and, on the other, fluctuations which make the net investment negative. Finally, concluding this part of the discussion, it will appear that, in the case of "forced savings," it is not only impossible to keep the rate of investment constant, but that there will exist, as a necessary consequence of the "forced saving," strong forces which tend to make the rate of investment negative.

II

Any lengthening of the process of production can only be completed over a period of time corresponding to the interval between the moment when the factors which are being shifted to an earlier stage are being invested, and the moment when their product matures. If the new, longer process is to be completed and maintained, this requires not only that the investment in the earlier stage must be constantly maintained, but also (except in a few rare cases, like the aging of wine and the growing of trees) that further complementary investments must be made in the later stages.

From this it follows that, in any progressive society, the particular forms in which investments are being made are determined by the expectation that, for some time to come, a similar stream of funds for investment will be forthcoming; and that, at any moment of time only a fraction of the funds available for new investment will be used to start new processes, while the rest will be required to complete the processes already under way. On the simplifying assumption that the *total* length of the marginal processes which are made possible by an increase in the supply of investible funds, is always greater than the total length of any process already used, this situation can be represented by the following diagram. The curvilinear[2] triangle ABC represents, in the same way as the triangle I have used in *Prices and Production*, the stock of capital belonging to processes already completed. (The area of the curvilinear triangle $AB'C'$ shows the stock of capital before the additions were begun.) The fully drawn stripes, beginning between C and D, represent incomplete processes, started at different moments in the past—and now in different stages of completion.

[2] The reasons which make a curvilinear triangle of the kind shown in the text a more appropriate representation than the simplified form used in *Prices and Production* are probably obvious.

The part of these stripes which is dotted represents the additional investment which is required to complete the processes. During every successive period of investment, part of the fund available will be used to start new processes, part to advance processes which are already under way, and part to complete the most advanced processes.

If at any moment, savings fall off by no more than had previously been used to start new processes, the completion of the processes al-

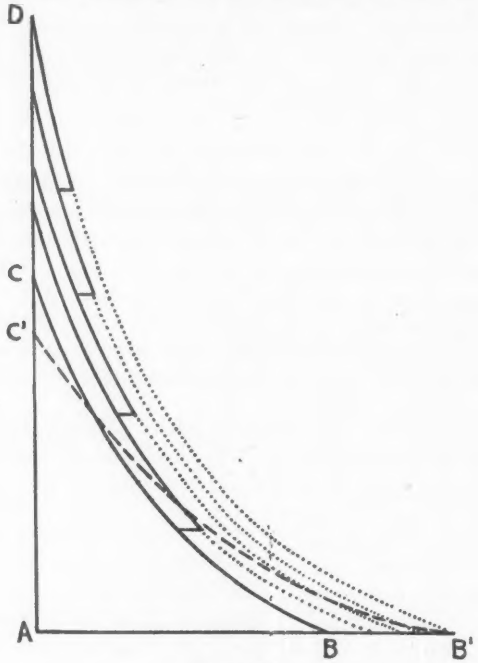

FIGURE I

ready under way will not be endangered. And since, at any moment, some of the unfinished processes will be completed, the amount of saving may continually fall off at a certain rate, and may reach zero at the moment when all the processes under way are completed. There is, therefore, at any moment, a maximum rate at which the rate of saving may fall off, without interfering with the new processes already started. It is only when the decrease in saving is faster than this rate, at which the need for capital for the purpose of completing processes under way decreases, that the incomplete structure cannot be com-

pleted and some of the investments made in the earlier stages have to be abandoned.

Such an abandonment of early stages will, of course, mean that the average period for which the current supply of original factors is being invested, is shortened even if, at the same time, a good deal of investment in new forms in later stages is taking place. In this case, however, investment in new forms need *not* mean *net* investment, since the losses on the abandoned earlier stages have to be offset against it.

III

It is only another way of stating the same conclusions if one says that the total demand for producers' goods will fall off in consequence of a decrease in the demand for new producers' goods only if the latter declines faster than the replacement demand increases, in consequence of the preceding growth of the stock of producers' goods. And this brings me to a discussion of the famous argument according to which any increase in the demand for productive equipment must lead to surplus capacity of plant producing that equipment, as soon as the demand for it ceases to increase. Although this is rarely recognized, this is a typical instance where an expansion of an earlier stage of production can be maintained only if the further increase of capital makes it possible to complete the structure by adequate increases of capital in the later stages.[3]

From the very outset, it is important in this connection carefully to avoid a confusion which arises from the failure to distinguish between fluctuations in the demand for productive equipment of a particular industry, which arise from fluctuations in the demand for the product of that industry, and fluctuations in the demand for new producers' goods in general, which are connected with fluctuations in the supply of funds available for new investment. Here I am mainly concerned with the latter type of fluctuation. How far what is to be said about this particular case is also applicable to the former, depends upon the degree to which the concrete capital equipment in earlier stages is specialized to the production of equipment for a particular industry, or whether it can be more generally used. On this question of fact, I can only refer to an interesting article which was recently published by Mr. Seltzer,[4] who seems to show that the mobility of capital in this

[3] A top-heavy structure of this kind is, therefore, an uncompleted structure in the sense that its earlier stages will be permanently employed only after they have helped to increase the equipment in later stages to such a magnitude that its replacement demand will fully use the capacity of the earlier stages. (The essential thing here, however, is not capacity in a technical sense, but sufficient employment to make amortization of plant at current prices possible.)

[4] "The Mobility of Capital," *Quarterly Journal of Economics*.

sense is far greater than is commonly supposed. Some other considerations on this point can be more appropriately discussed after I have dealt with the first type of fluctuation.

At first, therefore, I shall assume that the increase in investment is due to an increase in the supply of capital, and that the plant required to provide the new equipment is not adapted to the requirements of one industry only, but can be used fairly widely. The question then is whether, on the expectation of a continued growth of capital at about the same rate, it will appear profitable to expand the plant in the industries producing that equipment to a point where *any* decline in the current supply of new capital will make full use of that plant unprofitable.

The answer to this question is simply this: as long as the supply of capital does not decrease by more than the amount which has so far been used to construct the new plant[5] making that equipment, there is no reason why the demand for new equipment should fall off. In other words, the effect of a decrease in saving will simply be that the beginning of new round-about processes will be stopped but, if the decrease does not exceed a certain rate, there is no reason why the already existing plant should not be continuously used to add to the equipment in later stages. And as the replacement demand due to earlier additions to this equipment will continue to rise, the supply of new savings may even continue to fall at a certain rate without affecting the employment of the plant producing this equipment. The situation is, therefore, completely analogous to the case of fluctuations in the rate of saving already discussed: so long as the decline in the rate of saving does not exceed the amount which will permit the processes which

[5] It is assumed here that the construction of this additional plant for making the equipment in question can be carried out either with the help of similar plant already in existence, or of some other plant which can *also* be used to produce equipment for lower stages. This, of course, will always be the case since no capital is created without the help of some capital already in existence, which if it is a question of adding an earlier stage of production to those already existing, must, *ex definitione*, mean that these capital goods have hitherto been used in later stages. It may be mentioned here, since this has occasionally been a cause of confusion, that any given capital good need not, and usually will not, belong to any one given "stage" of production only. If it is used to produce other capital goods employed in different stages, and still more if it helps to produce durable goods, or is itself durable, it belongs to as many different "stages" as different periods of time elapse from the moment in which we consider it, to the moments when the different final products which it has helped to produce are consumed. This, however, so far from making the concept of stages useless, is only a necessary distinction in order to explain the different ways in which the value of individual capital goods will be affected by changes in the supply of capital, the rate of interest, or other factors affecting the structure of production.

have already been started to be completed, it need not have any harmful effects.

IV

The confusion on this point seems to result from a very common mistake—that of applying what is true of a single industry to industry as a whole. While, of course, the relative magnitude of the demand for equipment for a particular industry will depend upon the demand for the product of that industry, it is certainly not true to say that the demand for capital goods in general is directly determined by the magnitude of the demand for consumers' goods. While it is true that some contemporary economists have come so much under the influence of the under-consumptionist fallacy that they are prepared to say that the savings will never lead to a corresponding increase in investment, because they involve a decline in the demand for consumers' goods, and are therefore only a harmful and undesirable phenomenon, I certainly need not discuss this with economists who accept as much of my fundamental position as do Messrs Hansen and Tout. But, if one accepts the proposition that the magnitude of the total demand for producers' goods is not a simple derivative of the demand for consumers' goods, but that any given demand for consumers' goods can lead to methods of production involving very different demands for producers' goods, and that the particular method of production chosen will depend upon the proportion of the total wealth *not* required for immediate consumption, then we must certainly take the fluctuations in the supply of free capital, and not the fluctuations in the demand for consumers' goods, as the starting point for this kind of analysis.

There is, therefore, no reason to suppose that a general increase in the demand for new capital goods, which is due to an increase in the supply of saving, must lead to a decrease in the demand for capital goods, as soon as the rate of saving begins to decline. And since I am still abstracting from the case where investment is financed by the creation of credit ("forced saving") or any other purely monetary changes, it is difficult to see what factors can affect the *total* demand for new capital goods, other than the supply of savings. Only if we assume that changes in the rate of interest which can be earned on new capital lead to hoarding or dishoarding, would a new cause of change be introduced. But this is one of the cases of monetary changes in the demand for consumers' goods which I shall have to discuss later on.

On this point, my argument so far amounts to this: that in so far as we abstract from monetary changes, the demand for consumers' goods can only change inversely with the demand for producers' goods, and in consequence, so far from having a cumulative effect in the same direction as the latter, will tend to offset it in the opposite direction.

There is, however, still the case of mere shifts of demand between different kinds of consumers' goods which, of course, will have some effect on the demand for particular kinds of capital goods. An unexpected shift of this kind will undoubtedly have the effect that provision made for the supply of new equipment in the industry from which demand has turned away, will now prove excessive or, in other words, it will now become unprofitable to complete the longer processes in the expected way. But the total demand for new equipment will not be changed, and whether the equipment-producing plant already in existence will continue to be used or whether new plant will have to be built, will depend upon the technical considerations already mentioned.

V

So much for the pure, or barter, theory of the subject (in the sense of the usual assumption of theory that money exists to facilitate exchange but exercises no determining influence on the course of things, or, in other words, remains neutral—an assumption which is almost always made though not expressed in these terms). The discussion of the active influence which may be exercised by money in this connection is best begun with the peculiar effect of forced savings, which will lead us to another of the points of discussion, namely, the effect of monetary changes on the demand for consumers' goods. For the peculiar characteristic of forced saving, which distinguishes its effects from those of voluntary saving, is simply that it leads necessarily to an increase in the means available for the purchase of consumers' goods. For this reason, my thesis about the impermanence of capital accumulated by forced saving is directly bound up with my thesis as to the effects of a direct increase in the monetary demand for consumers' goods, which Messrs Hansen and Tout, quite consistently, also reject.

The reason why forced saving will always lead to a subsequent increase in the money available for the purchase of consumers' goods, is fairly obvious and will probably not be contested. Entrepreneurs are in this case enabled to attract factors of production from later to earlier stages, not by a corresponding transfer of funds from consumers' to producers' goods, but by additional money handed to them. This means that they will bid up the prices of these factors without there being a corresponding fall in the prices of other factors. Total money income will therefore increase, and this increase will in turn lead to an increase in the amount of money expended on consumers' goods. This increase in the expenditure on consumers' goods will necessarily follow in time upon the increase in the demand for factors. This lag will mean that, for some time after the demand for factors (or produc-

ers' goods) has ceased to increase (or when its rate of increase begins to slow down), the demand for consumers' goods will continue to increase at a faster rate; and so long as the increase in the demand for producers' goods is slowing down—and for some time afterwards—the monetary proportion between the demand for producers' goods and the demand for consumers' goods will change in favor of the latter.

The question turns, therefore, upon the effect of such a relative increase in the monetary demand for consumers' goods. The reply, in the particular case in question, however, is simplified, in comparison with the general problem, because, on our assumptions, two relevant points are given. We have in this particular case to assume that: (a) since it is the situation at the end of a boom, there are no unemployed resources, and (b) since the rate of credit expansion for productive purposes tends to be slowed down in spite of a continued rise in the monetary demand for consumers' goods, we cannot assume that the continued rise in this demand will lead to a renewed credit expansion. The much more difficult case of an increase in the monetary demand for consumers' goods, where these assumptions do not necessarily hold, as well as the problem why the rate of credit expansion cannot be sufficiently high to avoid this type of reaction, will be considered later on.

The relative rise in the price of consumers' goods will not only improve the competitive position of their producers on the market for original factors, but will also make it profitable for these to increase their output by the more rapid, even if more expensive, method of employing relatively more labour (original factors) in proportion to capital. And while their demand for all the non-specific factors of production (which can also be used in the latest stages of production) will continue to drive up the prices of these factors, the prices of the intermediate products specific to earlier stages of production will tend to fall relatively to their costs. And since the effect of this will not only tend to increase cumulatively towards the earlier stages but will also cause a shift of free capital towards the more profitable earlier stages, it is easy to see how more and more of the earlier stages will tend to become unprofitable, until unemployment finally arises and leads to a fall in the prices of the original factors of production as well as in the prices of consumers' goods.

VI

Before I turn to the aspects of the situation where unemployed factors and unused lending capacity of all banks exist, and where, perhaps, delay in making the necessary adjustments has led to prolonged unprofitability causing deflation and a rapid general fall of prices, a little more must be said about the rate of credit expansion which

would have to continue uninterruptedly if a reaction of the kind just discussed is to be avoided.

Messrs Hansen and Tout merely speak of a *steady* rate of credit expansion as a sufficient condition for a continuous and undisturbed rate of capital growth. I am not quite sure what "steady" means in this context. But if it refers, as is probably the case, to a constant rate of increase in the total media of circulation, I think it can be shown that this is not sufficient to maintain a constant rate of forced saving; while it seems that any attempt to make the rate of credit expansion great enough to secure a constant rate of forced saving will inevitably be frustrated by counteracting forces which come into operation as soon as the process of inflation exceeds a certain speed.

A constant rate of forced saving (i.e., investment in excess of voluntary saving) requires a rate of credit expansion which will enable the producers of intermediate products, during each successive unit of time, to compete successfully with the producers of consumers' goods[6] for constant additional quantities of the original factors of production. But as the competing demand from the producers of consumers' goods rises (in terms of money) in consequence of, and in proportion to, the preceding increase of expenditure on the factors of production (income), an increase of credit which is to enable the producers of intermediate products to attract additional original factors, will have to be, not only absolutely but even relatively, greater than the last increase which is now reflected in the increased demand for consumers' goods. Even in order to attract only as great a proportion of the original factors, i.e., in order merely to maintain the already existing capital, every new increase would have to be proportional to the last increase, i.e., credit would have to expand progressively at a constant *rate*. But in order to bring about constant additions to capital, it would have to do more: it would have to increase at a *constantly increasing rate*. The rate at which this rate of increase must increase would be dependent upon the time lag between the first expenditure of the additional money on the factors of production and the re-expenditure of the income so created on consumers' goods.

It is true that in *Prices and Production* I have not only not discussed in detail what rate of credit expansion is required to maintain a given rate of forced saving, but have simply assumed that that rate—

[6] I am compelled here—as I was in *Prices and Production*—to speak, for the sake of brevity, in terms of competition between the producers of intermediate products and the producers of consumers' goods (the present and future goods of Böhm-Bawerk's exposition), instead of speaking more correctly of competition between a continuous range of entrepreneurs in all "stages" of production, which leads to all original factors being invested for a shorter or longer average period.

whatever it was—could not be permanently maintained for institutional reasons, such as traditional banking policies or the operation of the gold standard. But I think it can be shown without great difficulty that even if these obstacles to credit expansion were absent, such a policy would, sooner or later, inevitably lead to a rapid and progressive rise in prices which, in addition to its other undesirable effects, would set up movements which would soon counteract, and finally more than offset, the "forced saving."

That it is impossible, either for a simple progressive increase of credit which only helps to maintain, and does not add to, the already existing "forced saving," or for an increase in credit at an increasing rate, to continue for a considerable time without causing a rise in prices, results from the fact that in neither case have we reason to assume that the increase in the supply of consumers' goods will keep pace with the increase in the flow of money coming on to the market for consumers' goods. In so far as, in the second case, the credit expansion leads to an ultimate increase in the output of consumers' goods, this increase will lag considerably and increasingly (as the period of production increases) behind the increase in the demand for them. But whether the prices of consumers' goods will rise faster or slower, all other prices, and particularly the prices of the original factors of production, will rise even faster. It is only a question of time when this general and progressive rise of prices becomes very rapid. My argument is not that such a development is *inevitable* once a policy of credit expansion is embarked upon, but that it *has to be* carried to that point if a certain result—a constant rate of forced saving, or maintenance without the help of voluntary saving of capital accumulated by forced saving—is to be achieved.

Once this stage is reached, such a policy will soon begin to defeat its own ends. While the mechanism of forced saving continues to operate, the general rise in prices will make it increasingly more difficult, and finally practically impossible, for entrepreneurs to maintain their existing capital intact. Paper profits will be computed and consumed, the failure to reproduce the existing capital will become quantitatively more and more important, and will finally exceed the additions made by forced saving.

It is important in this connection to remember that the entrepreneur necessarily and inevitably thinks of his capital in terms of money, and that, under changing conditions, he has no other way of thinking of its quantity than in value terms, which practically means in terms of money. But even if, for a time, he resists the temptation of paper profits (and experience teaches us that this is extremely unlikely) and computes his costs in terms of some index number, the rate of depre-

ciation has only to become fast enough, and such an expedient will be ineffective. And since the gist of my argument is that, for the purpose under discussion, the rate of credit expansion and depreciation has to increase at an increasing rate, it will in time reach any desired magnitude.

VII

For these reasons, it seems to me that the hope of Messrs Hansen and Tout based on a steady rate of forced saving is illusory. Whether there may not exist conditions under which temporary forced saving may take place without the evil consequences of a crisis, is quite another matter. That this will be possible only if the rate of forced saving is comparatively small, is probably obvious. Another condition which we already know is that the fluctuations in investment to which it gives rise keep well within the limits we have described. In another place,[7] I have tried to show that, if these conditions are combined with a third, namely, the presence of a relatively high rate of voluntary saving, which provides the means of taking over, as it were, the real capital which has been created but cannot be maintained by means of forced saving the loss of this capital may be avoided. But in this case, the only one I know where such a loss will be avoided, the forced saving will only mean an anticipation but no net increase of the circulation of capital, because it can only be maintained if an equivalent amount of saving is to be forthcoming later. For this reason, I am even more doubtful than before whether forced saving can ever be a blessing as Messrs Hansen and Tout think. This is quite irrespective of the question whether there is any sense in which the economist can legitimately say (as I have occasionally said myself) that such decisions made against the will of those concerned may be "beneficial." But that touches the much wider problem of whether we possess any gauge by which to measure the satisfaction derived by those concerned, except their own preferences, shown in their decisions—a question which I cannot even begin to discuss here.

VIII

It will be impossible within the compass of this article to discuss the further points made by Messrs Hansen and Tout in the same orbit as those more fundamental problems already taken up. Particularly, the next and very important point as to the effect of an expansion of consumers' demand at a time when the productive forces are not fully employed, and banks are in a position to expand credits to producers, could be answered completely only in connection with a fully

[7] "Stand und Zukunftsaufgaben der Konjunkturforschung," *Festschrift für Arthur Spiethoff*, p. 110.

developed theory of the process going on during a depression. But, if it be assumed that these two conditions exist as a consequence of a preceding crisis (and a definite assumption as regards the reason *why* these conditions exist is essential for any answer), and if the explanation of the crisis which I have just discussed is accepted, it is difficult to see how the same phenomenon, which has brought about the crisis, i.e., the rise in the relative demand for consumers' goods, should also be the cure for it. The scarcity of capital, which, of course, is nothing else but the relatively high price of consumers' goods, could only be enhanced by giving the consumers more money to spend on final products. At least so long as there are no further monetary complications, particularly so long as it is not assumed that the expectation of a further fall in prices has led to hoarding, I see no way of getting over this difficulty. But before I proceed to the relation between these secondary monetary complications, and the underlying real maladjustments which have caused it, I must try to clear away what seems to me to be a confusion which has led Messrs Hansen and Tout to apply their denial of the capital-destroying effect of additions to consumers' credits, not only to the peculiar situation of an advanced depression, but also generally.

The essence of the confusion on this point seems to me to lie in the contrast which my critics try to establish in several places between what they call "nominal" changes in the relative monetary demand for consumers' goods and producers' goods, and the "real changes in the demand for consumers' goods occasioned by a fundamental modification in time preference for present and future goods." It seems to me that, to assume that this rate of time preference can have any effect other than through the relative demand for these two classes of goods, or can have any immediate effects different from those of any other cause affecting that relative demand, is an attempt to establish a purely mystical connection. The mere fact that, even without a monetary change, any change in the distribution of the command over existing resources will, under a given set of individual time preferences, lead to quite different proportions between capital and income, should suffice to make this quite clear.[3]

Nor can I see how the two authors can combine their acceptance of the idea that forced saving can be brought about by monetary causes, without a change in the rate of time preference, with a general denial

[3] This fact is partly realized by the authors who, however, seem to underestimate its importance, mainly because they think only of the effects of a change in the distribution of *income;* and while this is obviously the only factor which will affect *new savings*, the total supply of free capital depends even more on turnover or amortization of existing capital. Any change in that stock of exist-

that monetary causes may also lead to "forced dis-saving." In principle, any change in the relative demand for the two categories of goods, whether brought about by actual shifts of monetary demand from one to the other, or merely by unilateral increases or decreases without corresponding changes on the other side, will *tend* to lead to corresponding changes in the relative amounts produced. The differences between these two cases (a shift and a unilateral change) are, *first* that the shift of an amount of money from the demand for consumers' goods to the demand for producers' goods changes the proportion between the two much more effectively than a mere unilateral increase or decrease by the same amount; and, *second*, that the changes in the quantity of money, which are implied in the second type of change, will lead to further changes which may counteract or offset the tendency created by the change in relative demand. This will be particularly true if a change in relative demand is accompanied by an absolute reduction in demand and if, at the same time, costs (i.e., the prices of the original factors of production) are rigid. In this case, deflationary tendencies are likely to set in, which may more than counter-balance the effect of the changed relative demand. But, in spite of these further complications which seem likely to arise, the principle seems to me to be true, and to comprise even what seems to Messrs Hansen and Tout a *reductio ad absurdum* of the argument, namely that a unilateral decrease in the demand for consumers' goods may lead to a lengthening of the structure of production. Although I fully admit that, because of the probable complications, this case is very unlikely to materialize, I do not think that it is entirely impractical. Would Messrs Hansen and Tout deny that, e.g., increased hoarding on the part of a class of very small rentiers who reduced their consumption of agricultural products, might not lead, via the reduction of wages, first in agriculture and then generally, to an increase in the real quantity of labor corresponding to a constant amount of money invested in industry, and therefore of capital?

IX

The analysis of this and similar cases would help to bring out an important distinction which Messrs Hansen and Tout tend to over-

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ing capital, brought about by monetary causes, will, by means of the consequent redistribution of the command over resources, tend to affect the relative demand for producers' and consumers' goods. If the monetary causes have lead to a destruction of capital, this change will necessarily be permanent. If they have led to the creation of additional capital goods, the effect on relative demand *may* be, at least to some extent, permanently to increase the relative demand for capital goods.

look: the distinction between the tendencies set up directly by a given monetary change, and the effects of the further monetary changes which may, and perhaps even probably will, but need not, be induced by this first change. A sharp dividing line is the more necessary here since the tendency in current discussions is either to take these secondary monetary changes for granted, without ever mentioning them, or to fail to demonstrate why, and under what conditions, they should follow the first change.

These considerations bring me back to the problem of the relation between the demand for consumers' goods and the prices of capital goods. I should not deny that there may be conditions where, e.g., the expectation of a general price fall has led to extensive hoarding, and where any change in this expectation may lead to such dishoarding of funds available for investment as to outbalance the initial effect of the increase in the demand for consumers' goods.[9] Nor is it inconceivable that a similar situation may prevail as regards bank lending. There can also be no doubt that, in connection with these secondary monetary complications, *general* price movements, apart from the changes in relative prices, will be of the greatest importance, and that anything which stops or reverses the general price movement may lead to induced monetary changes, the effect of which on the demand for consumers' goods, and producers' goods, may be stronger than the initial change in the quantity of money.

But one has to be careful not to fall into the error apparently made by Messrs Hansen and Tout—that of assuming that, in all cases, where the prices of consumers' goods and producers' goods move in the same direction (e.g., upwards), this may not be accompanied by changes in their relative height, which would produce exactly the same effect as if there were no general price movement. Their general proposition that changes in the relative prices of consumers' goods and producers' goods will not have the same effect when they are accompanied by a universal movement in the same direction as when they find expression in an absolute movement in different direction, is only true under the following assumptions: (1) that the expected general price movement is relatively great compared with the relative price changes; (2) that, at the same time, the general movement does not exceed the limits beyond which—as experience has shown at least in cases of considerable inflation—costs begin to move more rapidly than prices; (3) that money rates of interest do not adapt themselves to the expected rate of general price change.[10]

[9] Cf. my contribution to the *Festschrift für Arthur Spiethoff*, quoted above.

[10] It is a curious fact that the discussion of the supposedly different effect of changes in the relative demand which are due to changes in the supply of money

Further, it is necessary to be careful to make clear the special assumptions under which these further complications are likely to arise. The deflationary tendencies, which are assumed to exist in most of the reasoning of the kind discussed, are not a necessary consequence of any crisis and depression, but are probably due to resistances to the necessary readjustments, caused by rigidity of prices, the existence of long term contracts, etc. I am far from underrating the importance of these phenomena. What I am pleading for is only that, for analytical purposes, these tendencies should carefully be kept separate and not confused with one another. Only in this way can we hope ultimately to unravel the tangle of different forces at work during a depression, and to arrive at that detailed explanation of the depression which I cannot even attempt here. But to deny the existence of certain tendencies merely because they are likely to be counteracted by others, does not seem to me to be a promising procedure.

X

The objections raised by Messrs Hansen and Tout to what they call my theses 7, 8, 9, and 10, are partly based on arguments which I have already discussed and partly introduce further complications which any program of practical policy has to face and which I admit I have not investigated sufficiently. But it is obviously impossible to develop my ideas further, in this connection, or to try to make good these deficiencies here.

There are only two more points upon which I wish to touch. The first is that the concept of neutral money was meant in the first place to be an instrument of theoretical analysis and not necessarily a tool of practical policy. Its purpose was to bring out clearly the conditions under which we could expect the economic process in a money economy to correspond perfectly to the picture drawn by the theory

leads the two authors to argue—in effect, if not explicitly, and on what seems to me to be wrong grounds—what they had previously denied, namely that capital accumulated by means of "forced saving" will not be permanent. If it were true that when, after a change in the supply of money, "equilibrium is finally established, the relation (between the prices of consumers' goods and the prices of producers' goods) will be found unaltered unless the effects of the transition period have been such as to change permanently the time preference of the income receivers" (p. 143), then, no doubt, the greater part of the real capital created by means of forced saving would be lost. But I think it will be clear by now why I should be very reluctant to use this argument in defence of my position.

Messrs Hansen and Tout think that such a permanent change in the time preference of the income receivers "is not unlikely, since an increase or decrease in money supply is likely to increase the real income of the community." This seems to show conclusively that what they have in mind is not the effect on the quantity and distribution of resources, but on individual time preferences.

of equilibrium and, incidentally, to show what we should have to consider as the peculiar active effects caused by monetary changes. In a sense, of course, this would also set up an ideal of policy. But it is by no means inconceivable that considerations other than the direct monetary influences on prices, such as the existence of long term contracts in fixed sums of money, the rigidity of prices, and such like institutional factors, may make such an attempt entirely impracticable, because it would set up frictions of a new kind. In that case, the task of monetary policy would be to find a workable compromise between the different incompatible aims. But, in this case, one would have to be clear that certain important determining and disturbing influences arising from monetary causes would remain in existence, and that we should always have to remain conscious of this fact. Or, in other words, that even under the best practicable monetary system, the self-equilibrating mechanism of prices might be seriously disturbed by monetary causes.

The second point is that up to 1927 I should, indeed, have expected that because, during the preceding boom period, prices did not rise—but rather tended to fall—the subsequent depression would be very mild. But, as is well known, in that year an entirely unprecedented action was taken by the American monetary authorities, which makes it impossible to compare the effects of the boom on the subsequent depression with any previous experience. The authorities succeeded, by means of an easy-money policy, inaugurated as soon as the symptoms of an impending reaction were noticed, in prolonging the boom for two years beyond what would otherwise have been its natural end. And when the crisis finally occurred, for almost two more years, deliberate attempts were made to prevent, by all conceivable means, the normal process of liquidation. It seems to me that these facts have had a far greater influence on the character of the depression than the developments up to 1927, which from all we know, might instead have led to a comparatively mild depression in and after 1927.

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A COMPARISON BETWEEN DIFFERENT DEFINITIONS OF COMPLEMENTARY AND COMPETITIVE GOODS

By R. G. D. Allen

The relationship between a given pair of consumers' goods can assume various forms in an individual's scale of preferences. It is usual, however, to allot the pair of goods to one or other of two main classes, the classes of complementary and of competitive goods. A number of criteria or definitions of the distinction between complementary and competitive goods have been suggested, and of these the most precise, from the mathematical standpoint, are the simple criterion first put forward by Edgeworth and Pareto and the more complicated one given in a recent paper by the present author.[1] The object of these notes is to examine further both these definitions and, in particular, to discover what connection there is between them.

The definition of complementary and competitive goods given by Edgeworth and Pareto assumes the existence of a total utility function which gives, for the individual under consideration, the utility of any combination $(x, y, z, \cdots)$ of the set of consumers' goods $X, Y, Z, \cdots$, appearing in his scale of preferences. Denoting this function by $u=\phi(x, y, z, \cdots)$, the relation between any pair of goods X and Y depends on the sign of the second order derivative $\dfrac{\partial^2 u}{\partial x\,\partial y}=\phi_{xy}$. If ϕ_{xy} is positive, so that the marginal utility of X increases when the amount possessed of Y increases and conversely, the goods are called complementary. If ϕ_{xy} is negative, on the other hand, the marginal utility of X decreases when the amount possessed of Y increases and conversely, and the goods are called competitive. This Edgeworth-Pareto definition, however, ignores two vital points and fails in its object. First, even if the existence of a total utility function can be assumed, the function is not to be taken, in general, as unique. Pareto himself admirably emphasized this point but appeared to overlook the logical corollary that the form and the sign of the derivative ϕ_{xy} are not determinate. Without further investigation, it is not possible, therefore, to use the derivative in a precise definition of complementary and competitive goods. Second, even if ϕ_{xy} can be made determinate, its value and sign vary according to the "position" of the individual, i.e.,

[1] See Edgeworth, "La teoria pura del monopolio," *Giornale degli Economisti*, 1897; Pareto, *Manuel d'économie politique* pp. 572–578; Allen, "Nachfragefunktion für Güter mit korreliertem Nutzen", *Zeitschrift für Nationalökonomie*, January 1934.

according to the amounts of the various consumers' goods he happens to possess. There is no reason whatever why ϕ_{xy} should not be positive at one position and negative at another. It would seem either that Edgeworth and Pareto intended their definition to apply only in the special cases when ϕ_{xy} preserves a uniform sign at all positions, or that they allowed a pair of goods, for a given individual, to be complementary in one set of circumstances and competitive in another. In the former case the definition loses in generality, while in the latter case it does not fit in with our everyday notion of the meaning of the terms complementary and competitive goods. The Edgeworth-Pareto criterion is, therefore, inconvenient and there are strong reasons for seeking a better definition.

In order to avoid the two difficulties mentioned above, I elaborated a distinction first made by Johnson, and introduced a different definition. Here the criterion of complementary and competitive goods is designed to be independent both of the existence of a utility function and of the indeterminateness in the function, if it can be assumed to exist. Further, it takes full account of the fact that the relationship between a given pair of goods in the individual's scale of preferences can vary in nature under different sets of circumstances. The definition only assumes that the individual's scale of preferences can be represented by the ratios of continuous marginal utility functions. If $\phi_x(x, y, z, \cdots)$, $\phi_y(x, y, z, \cdots)$, $\phi_z(x, y, z, \cdots)$, . . . are the marginal utilities of the goods X, Y, Z, $\cdots$ respectively, then the important expression in the specification of the relation between any pair of goods X and Y is $m = -\dfrac{\phi_x}{\phi_y}$. This indicates, at any position, the individual's "direction of indifference" with respect to X and Y.[2] The position of the individual is defined as "normal" relative to the pair of goods if $\dfrac{\partial m}{\partial x}$ is positive and $\dfrac{\partial m}{\partial y}$ negative. The position is "non-normal" if one of these inequalities is reversed,[3] so that either $\dfrac{\partial m}{\partial x}$ and $\dfrac{\partial m}{\partial y}$ are both negative, in which case X is the "superior" and Y the "inferior" good, or the expressions are both positive, X being the in-

[2] The numerical value of m, which is usually negative, gives the ratio of an (infinitesimal) amount of Y to an (infinitesimal) amount of X either of which would just compensate the individual, at the given position, for the loss of the other. If a utility function and indifference curves (for X and Y) exist, m represents the slope of the tangent to the indifference curve through any point.

[3] It is not possible to reverse both inequalities by reason of the assumptions of the problem (see Allen, *op. cit.*).

ferior and Y the superior good. Finally, the goods X and Y are *complementary* if they are normally related at all positions, and *competitive* if they are non-normally related at some (but not necessarily at all) positions.

The significance of normal and non-normal positions in the relationship between a pair of goods can be expressed in two ways. At a normal position, an increase in the amount possessed of one good (it does not matter which) decreases the individual's marginal utility of this good relative to that of the other. But, at a non-normal position, the marginal utility of the good that is the superior of the pair must increase relative to that of the inferior good for any increase whatever in the individual's possession of the goods. Alternatively, if the individual increases his possession of one of the goods from a normal position, he finds that he then requires more of the increased good to compensate for a given loss of the other good than he did before the increase. From a non-normal position, however, the same result follows any increase in the amounts the individual possesses of the two goods—more of the inferior good is always required, after an increase, to compensate for a given loss of the superior good. Roughly, a normal position is characterized by the fact that either good loses in relative importance when the individual gets more of it, while, at a non-normal position, it is always one definite good (the inferior) that loses in relative importance whenever the individual increases his possession of the goods in any way.

What effect has the sign of ϕ_{xy} (used in the criterion of Edgeworth and Pareto) on the complementary or competitive nature of the relationship between the goods X and Y defined in this way? To investigate this question, our general definition must be reduced and considered in the important particular case where a utility function exists (with ϕ_{xy} as a second order derivative). Taking, for convenience, the case of two goods X and Y only, it is assumed that a continuous utility function exists, the two first order partial derivatives of which are proportional to the marginal utility functions. This function is not unique; it is simply an index, and not a measure, of utility. Let one of its forms be $u=\phi(x, y)$ and arrange the marginal utilities, ϕ_x and ϕ_y, so that they are the actual derivatives of ϕ. It can be assumed, further, that the indeterminateness in the utility function does not affect the signs of $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$, which can be represented by ϕ_x, ϕ_y, ϕ_{xx} and ϕ_{yy} respectively. This involves little loss of generality.[4] But the other

[4] This statement can be made more precise. The indeterminateness in the utility function expresses itself in the general form: $u=F\{\phi(x, y)\}$ where F is an arbitrary function. The corresponding general expressions for the derivatives are

second order derivative, $\dfrac{\partial^2 u}{\partial x\,\partial y}$, cannot be taken as determinate in sign; it is affected by the form adopted for the utility function. In particular, ϕ_{xy} is only one form of this derivative and no definite importance can be attached to its sign. There is, however, one important exception to this assertion. The sign of $\dfrac{\partial^2 u}{\partial x\,\partial y}$ is quite definite, and ϕ_{xy} can be used to represent it, when either ϕ_x or ϕ_y happens to be zero.[5]

As regards the derivatives with definite signs, the following conditions can be imposed. First, as an expression of the "law" of decreasing marginal significance, ϕ_{xx} and ϕ_{yy} can always be taken as negative. This implies that the marginal utility functions decrease, i.e., that ϕ_x decreases as x increases and ϕ_y decreases as y increases.[6] Further, it can be taken that ϕ_x is positive (and decreasing) for values of x up to a definite "saturation" value, where ϕ_x becomes zero. The saturation value of x is the solution of $\phi_x = 0$ and must depend on the value of y that is given in the derivation of ϕ_x. Denote the saturation value by $x = \bar{x}(y)$. In the same way, ϕ_y is positive, decreases and becomes zero at the saturation value $y = \bar{y}(x)$. The region of the plane OXY bounded by the co-ordinate axes and the saturation curves, $x = \bar{x}(y)$ and $y = \bar{y}(x)$, can be called the "effective region." At all positions within this region, the marginal utilities, ϕ_x and ϕ_y, are both positive, except on the curved boundaries where one or the other vanishes.

As far as the relation between the goods is concerned, the effective region need only be considered and, in fact, the definition of complementary and competitive goods adopted above is implicitly restricted

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$$\frac{\partial u}{\partial x} = F'(\phi)\cdot\phi_x \qquad \frac{\partial u}{\partial y} = F'(\phi)\cdot\phi_y$$

$$\frac{\partial^2 u}{\partial x^2} = F'(\phi)\cdot\phi_{xx} + F''(\phi)\cdot\phi_x^2 \qquad \frac{\partial^2 u}{\partial y^2} = F'(\phi)\cdot\phi_{yy} + F''(\phi)\cdot\phi_y^2.$$

Since ϕ_{xx} and ϕ_{yy} are taken below as negative, these derivatives are determinate in sign if (for example) the form of F is restricted so that $F'(\phi)$ is positive or zero and $F''(\phi)$ is negative or zero.

[5] In the notation of the previous footnote,

$$\frac{\partial^2 u}{\partial x\,\partial y} = F'(\phi)\cdot\phi_{xy} + F''(\phi)\cdot\phi_x\cdot\phi_y,$$

and this expression varies with the form assumed for $F(\phi)$. But, when either ϕ_x or ϕ_y is zero, $\partial^2 u/\partial x\partial y = F'(\phi)\cdot\phi_{xy}$, an expression which is completely determinate in sign.

[6] It is quite possible that this condition of decreasing marginal utility fails to hold in certain cases of related goods. This possibility is not considered here in connecting the general criterion of complementary and competitive goods with that of Edgeworth and Pareto. The general criterion itself is, of course, quite independent of the condition.

to this region. But attention can be particularly directed to those positions lying on the curved boundaries of the effective region, and this for two reasons. It has already been noticed that ϕ_{xy} becomes determinate in sign when either ϕ_x or ϕ_y vanishes and, therefore, *it is to positions on the boundaries that we must look for any connection between* ϕ_{xy} *and the criterion of complementary and competitive goods.* Further, at positions on the boundaries, the individual finds himself "saturated" with one of the goods while his possession of the other good can be relatively small. These positions are particularly suitable for throwing into prominence the complementary or competitive nature of the relationship between the goods. In the case of complementary goods, it can never happen that one good becomes superior to the other even when the individual is nearly "saturated" with the latter. On the other hand, it will be seen that one of a pair of competitive goods is to be expected to become superior to the other just at these "saturation" positions.

Now,

$$\frac{\partial m}{\partial x} = -\frac{\phi_{xx}\,\phi_y - \phi_{xy}\,\phi_x}{\phi_y^2}$$

$$\frac{\partial m}{\partial y} = \frac{\phi_{yy}\,\phi_x - \phi_{xy}\,\phi_y}{\phi_y^2}$$

these expressions being independent of the indeterminateness in the utility function. At normal positions, there is no information about the derivative ϕ_{xy}; it is indeterminate and can be either positive or negative in sign. But at non-normal positions, where either $\frac{\partial m}{\partial x}$ is negative or $\frac{\partial m}{\partial y}$ positive, it follows that ϕ_{xy} must be determinate in sign and must, in fact, be negative. The only general fact we have, therefore, is that a negative sign of ϕ_{xy} is a characteristic of non-normal positions; it is a necessary, but not a sufficient, condition for such a position.

But more information can be obtained at positions on the curved boundaries of the effective region. At all positions on the boundary where the individual is saturated with the good X, ϕ_x vanishes and

$\frac{\partial m}{\partial x} = -\frac{\phi_{xx}}{\phi_y}$ having a positive sign;

$\frac{\partial m}{\partial y} = -\frac{\phi_{xy}}{\phi_y}$ having the opposite sign to ϕ_{xy}.

The relationship between X and Y is now completely fixed by the sign of the (determinate) derivative ϕ_{xy}. A position on the X-saturation boundary is normal if, and only if, ϕ_{xy} is positive, and it is non-normal

if, and only if, ϕ_{xy} is negative. Further, all the non-normal positions that occur on this boundary have X as the inferior and Y as the superior good. Exactly the same results hold on the Y-saturation boundary, except that X is the superior and Y the inferior good at non-normal positions. By continuity of the functions concerned, what holds on the boundaries themselves also holds at positions sufficiently near the boundaries. Hence, though ϕ_{xy} is not sufficient to determine the relation between X and Y within the effective region, there is a definite connection on and near the boundaries of the region. The goods are normally related on and near either boundary only if ϕ_{xy} is positive, and non-normally related only if ϕ_{xy} is negative. Further, if non-normal positions occur near the X-saturation boundary, Y is the superior of the two goods, and conversely.

To return to the general relation between the pair of goods, if X and Y are complementary goods, then their relationship must be normal at all positions, even at those on or near the saturation boundaries. *For complementary goods, ϕ_{xy} must be positive at some positions*, i.e., all positions on or near the boundaries of the effective region. On the other hand, if X and Y are competitive goods, then there must exist some positions at which their relation is non-normal, and at these positions ϕ_{xy} is negative. *For competitive goods, ϕ_{xy} must be negative at some positions.* These results are independent of the indeterminateness of the utility function. It follows at once that, in the special cases where the utility function exists in a unique form and where ϕ_{xy} is positive (negative) at all positions, the pair of goods must be complementary (competitive). The two definitions of complementary and competitive goods are, therefore, consistent in these special cases and the general criterion need not be used since the simpler criterion of Edgeworth and Pareto serves equally well. For cases other than these two, however, there is now no doubt that the Edgeworth-Pareto criterion is insufficient.

It is not necessary, in the case of competitive goods, that the non-normal positions (at which ϕ_{xy} is negative) should be located near the boundaries of the effective region. All that is necessary for competitive goods is that the effective region should contain, somewhere within it, at least one non-normal position. But it can be reasonably maintained that, for any definitely competitive pair of goods, the non-normal positions are clustered near (or on) the boundaries of the effective region, those occurring near the boundary where X is the saturated good having Y superior, and conversely.[7] In fact, the test of com-

[7] Further, it can be maintained that the non-normal positions are more likely to occur clustered about those parts of the boundaries of the effective region near the co-ordinate axes, i.e., well away from the "complete saturation point"

petitive goods is usually that one good can become superior to the other whenever the individual possesses a relatively small amount of the former and is almost saturated with the latter. Granting this assertion, it is possible to distinguish two broad classes of competitive goods, classes which shade off one into the other:

Class A: where all, or nearly all, the non-normal positions occur near one boundary only. In the extreme case here, the relation between the goods ceases to be normal only when the individual is "saturated" with one definite good of the pair and this good must always be inferior to the other whenever the relationship is non-normal. This class of competit. e goods is characterized by an unsymmetrical relation, one of the pair being definitely superior and the other definitely inferior.

Class B: where the non-normal positions are located more or less evenly near both boundaries. Here, each good can become superior when the individual is nearly "saturated" with the other. This can be called the class of mutually competitive goods.

SUMMARY

If the utility function can be defined in a unique manner, with respect to a pair of goods X and Y, so that the second order derivative ϕ_{xy} is either positive at all positions or negative at all positions, then the goods are complementary or competitive respectively. In these two clear-cut cases, the definition of complementary and competitive goods framed by Edgeworth and Pareto serves just as well as the more complicated definition indicated here. But the relationship between X and Y need not appear in these well-defined ways, and it can assume much more involved forms. The Edgeworth-Pareto criterion must then break down owing to the indeterminateness in the form and in the sign of ϕ_{xy}, and the definition suggested here must be used. Even if the derivative ϕ_{xy} exists at all, it only becomes free of indeterminateness on and near the boundaries of the effective region, where one of

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in which the boundaries intersect. The non-saturated good is then possessed in relatively small quantities. The results, indicated here as "reasonable," certainly hold for a quadratic utility function. If

$$\phi(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy$$

and if h is negative, then $\phi_{xy} = 2h$ is negative at all positions. The goods are, therefore, definitely competitive (on both criteria). It can be easily shown that the non-normal positions, with Y as the superior good, occur at all points of the effective region lying between a line perpendicular to OX and the X-saturation boundary, which is, in this case, a negatively sloped straight line. A similar result holds for non-normal positions which have X as the superior good. Hence, the non-normal positions are all located near the boundaries and are more frequent near the co-ordinate axes.

the marginal utility functions becomes zero and where the individual is "saturated" with one or other of the goods. Now, the relationship between a pair of goods is particularly well marked at positions near these boundaries. Complementary goods must remain "normally" related even when the individual is nearly saturated with one of the goods. On the other hand, one of a pair of competitive goods can, and usually does, become "superior" to the other at many of these saturated positions. In terms of ϕ_{xy}, the characteristic of complementary goods is that ϕ_{xy} is positive at all positions on and near the boundaries of the effective region, while competitive goods are indicated by ϕ_{xy} assuming negative values which occur (apart from odd cases) on and near one or both boundaries.[8]

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[8] Since this article was written (August 1933) the problem of complementarity has received further consideration in a joint article by Dr. J. R. Hicks and the present writer, "A Reconsideration of the Theory of Value," *Economica*, February and May, 1934.

INTERDEPENDENCE OF CYCLICAL AND SEASONAL VARIATION

By J. Wiśniewski

Paper read before the meeting of the Econometric Society, Leyden, September 30, 1933

The problem I propose to discuss is novel still, one that has not yet received the attention it deserves. As far as I know, only the following students have worked on it: Åkerman of Sweden, Gjermoe of Norway, and Kuznets of the United States.[1] What I consider to be my contribution is, first, a statistical study of Polish data, second, a linking together of statistical evidence with business cycle theory.

For the first part of my work I adopted *in toto* the method advanced by Kuznets. This method consists in correlating (a) the deviations of seasonal variation indices considered as standard, calculated by the ratio-to-moving-average or some other method, denoted by s and (b) the percentage deviations of the observed values from their thirteen months' moving average, the first and thirteenth month each being weighed at a half, denoted by d. A separate calculation is made for each year. Thus we get a coefficient of correlation r_{sd} measuring the degree of similarity between the seasonal pattern for the given year and that taken as normal, and a coefficient of regression, b_{ds}, of d on s, which is a measure of seasonal amplitude. The latter measure proved rather the more useful in the subsequent study.

My study covered eleven series derived from various branches of economic activity, but for the sake of brevity I will speak only of those relating to employment in manufacture in Poland. After the b's were calculated, I proceeded to the analysis of their behavior in respect to cyclical variation. The first working hypothesis I set was that the b's should approximate a decreasing function of the level of employment in the given series. When the total volume of business done in a certain year is small, the tendency of manufacturers is to concentrate it in a short span of time, viz., when the seasonal conditions of production are best. A striking instance of such a tendency is shown by the employment in the Polish beet sugar industry (Note Fig. 1). The total volume of production in this industry is done during the months from October to January inclusive, but the most convenient months are October and

[1] The Polish Institute for Economic Research, in complete independence of the writers mentioned, has stressed since 1931 the phenomenon of "pseudo-seasonal" variations, i.e., more pronounced seasonal variations in depression years.

November. A glance at the chart shows that the smaller the total volume of production, the more is the employment concentrated in the

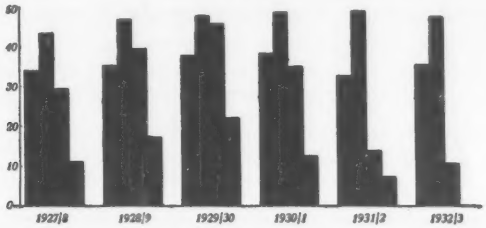


FIGURE 1.—Employment in the sugar industry—raw data, thousands of workers employed. The bars for each year indicate employment in October, November, December, and January.

last-mentioned months. In October and November, the changes of employment from year to year are much less violent than in the two other months, especially January.

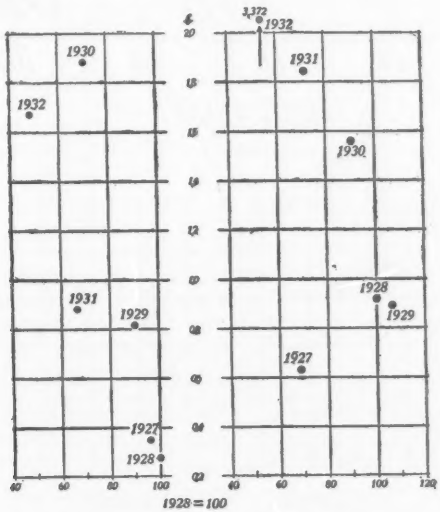


FIGURE 2.—Employment

As the volume of business and employment becomes concentrated in a shorter part of the year, the amplitude of seasonal variation is

bound to expand; during the phase of prosperity the converse takes place. Gjermoe tried to express this relation in a form of a regression equation (the particular form chosen by him was that of a hyperbola), but the several points do not cluster very closely around the curve.

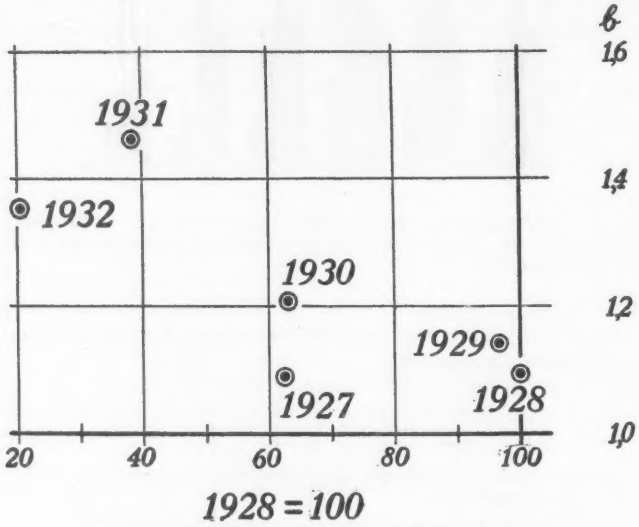
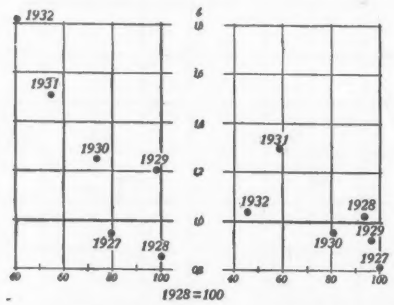


FIGURE 3.—Employment in the building industry



Minerals Timber

FIGURE 4.—Employment

Nor do they in the examples worked out by me. While I refrained from calculating definite regression curves, I prepared for each of the series analyzed a scatter diagram, where the level of employment in the given

series (expressed as a percentage of 1928), is shown on the axis of the abscissae, b being the ordinate (Figs. 2–4).

In most cases the relation mentioned seems to be fairly clear for the years 1928 to 1931, while 1927 and 1932 give considerably smaller values of b than could be expected; 1929 gives sometimes rather large values because of the severe winter conditions then prevailing. The explanation of the discrepancy of the behavior of the b coefficient can only be made in connection with business cycle theory, and especially with that formulated by my colleague Kalecki.[2]

As a matter of fact, the force governing the amplitude of seasonal variation in manufacture is not the level of production itself but its ratio to the productive capacity.

Now my scatter diagrams, as well as Gjermoe's regression equations, were constructed with the hope of proving that the b's are a decreasing function of the level of employment and production. This relation may be realized only so far as the level of production moves in the

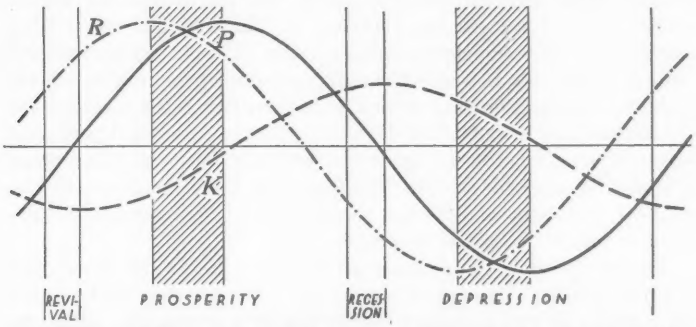


FIGURE 5.—Scheme of the business cycle. P—production, K—productive capacity, R—ratio of P to K. It is not proposed to represent absolute values of those magnitudes, but the direction of their changes only. The division into phases is done accordingly to Mr. Kalecki's theory.

same direction as its ratio to the productive capacity. In the light of Mr. Kalecki's theorems we can note when this condition is fulfilled. (Even if one does not accept his theory in full, his description of the behavior of the several curves seems to be realistic.) The ratio of production to capacity precedes the curve of production by about one-eighth of the length of the cycle (see Fig. 5). So we may expect a disagreement between the movements of the two curves by such a time before the upper and lower turning points of the curve of production,

[2] I am indebted to Mr. Kalecki for constructive criticism of the present communication.

and correspondingly a discrepancy between the actual values of the b coefficient and its values read from a drawn or estimated curve of regression on the production index. That is precisely what we observe in our Polish data for the year 1932, i.e., one year before the lower turning point of the production curve, which happened to fall in 1933. If we take into account that the length of the cycle amounts to about 8–10 years, one year is just $\frac{1}{8}$ of a cycle length, and the agreement between theory and fact in this point seems to be practically perfect. If this agreement were more widely proved, the behavior of seasonal amplitude coefficients could be considered as a forecaster of turning points in the general business curve.

Aside from the normal cyclical variation, other causes may contribute to produce abnormal values of the b coefficients, such as violent structural changes or an abnormal prolongation of cyclical depression. The former elements were at work in Poland up to 1927.

This ends the main argument of my communication. Some minor points might, however, be touched on. One has to do with the so-called seasonal pattern i.e., the shape of the seasonal curve. This, measured by the coefficient r, undergoes but little change as compared with b. Sometimes, however, the changes are significant and can be explained in connection with cyclical variation. This is the case with the textile industry in Poland. This industry has two principal periods of productive activity: one for production of fabrics for spring and summer clothing, another for winter. The former is much more sensitive to cyclical variation, and so the seasonal pattern undergoes definite changes with the course of general business.

Another point is the dependence of the cyclical variation on the seasonal one. Mr. Åkerman stresses this point very much, but he makes no attempt at any empirical verification of this relation. What we speak of is the hypothesis that cyclical changes occur more frequently and more violently at some specific season. Åkerman says this is the fall, the stimulus coming from the harvest. I am inclined to think that the period when the volume of business is determined for one year is in the spring, in connection with the building season. If construction is bound to shrink, the shrinkage is felt during winter and early spring when jobs are finished and no new ones are started. If there is an expansion it starts in spring. As far as quantitative analysis is concerned, I tried various methods but no one gave quite satisfactory results. It seems as if the two ways of interdependence were not capable of being untangled. Very frequently we cannot say whether the cyclical variation is the cause of the seasonal change or vice versa.

Polish Institute for Economic Research

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7. Jan Wiśniewski, "Wahania sezonowe w budownictwie"—"Les fluctuations saisonnières dans l'industrie du bâtiment," *Kwartalnik Statystyczny*, Warsaw, 1933, No. 2, pp. 239–296. A French summary is added.

TABLE I

EMPLOYMENT IN THE POLISH MANUFACTURING INDUSTRY: UNADJUSTED FIGURES, THOUSANDS OF WORKERS'-HOURS PER WORKING DAY

Year and month		Textiles	Apparel	Building	Minerals	Timber
a1926	VII	710	37	187	334	337
	VIII	836	46	188	341	315
	IX	949	51	198	331	306
	X	1004	59	195	319	305
	XI	999	60	183	289	315
	XII	987	57	134	276	315
a'27	I	1029	56	104	259	325
	II	1078	57	94	268	370
	III	860b	64	119	278	387
	IV	1061	64	142	343	375
	V	1142	71	178	425	411
	VI	1111	69	185	428	406
	VII	1088	67	210	435	409
	VIII	1189	72	257	439	382
	IX	1206	75	277	424	357
	X	1191	80	287	404	334
	XI	1209	82	224	379	327
	XII	1156	76	142	361	328
'28	I	1199	79	134	334	346
	II	1239	93	145	360	399
	III	1219	98	169	394	430
	IV	1092	97	232	445	411
	V	1150	101	313	502	422
	VI	1107	100	368	547	432
	VII	1057	94	364	540	418
	VIII	1200	107	403	555	404
	IX	1190	113	391	535	391
	X	919b	119	396	482	356
	XI	1267	113	372	458	367
	XII	1238	103	262	421	342
'29	I	1258	100	171	364	358
	II	1203	102	139	324	358
	III	1174	105	167	358	387
	IV	1131	111	244	432	415
	V	997	112	328	525	418
	VI	947	110	363	564	434

The following adjustments have been introduced:

a. The figures for 1926 and 1927 are not those really observed. As the number of reporting firms increased since January, 1928, figures for months prior to this date had to be multiplied by adequate adjustment factors, not exceeding 1.1.

b. In the period in question, three major strikes occurred in the textile industry. In the computation of moving averages the following numbers were taken instead of those observed: 1070 for March, 1927; 1200 for October, 1928; 635 for March, 1933.

c. Because of the abrupt decrease in employment figures at the end of 1929, the values of the moving average for September to November, 1929, has been taken as 976.1 and for January to May, 1930, as 822.4.

d. At the time the computations of the b and r coefficients were made, the figures for May and June, 1933, had not yet been published. Small discrepancies may result from the extrapolations made.

TABLE I (*Continued*)
EMPLOYMENT IN THE POLISH MANUFACTURING INDUSTRY: UNADJUSTED FIGURES, THOUSANDS OF WORKERS'-HOURS PER WORKING DAY

Year and month		Textiles	Apparel	Building	Minerals	Timber
	VII	884	101	363	553	411
	VIII	1027	110	371	562	391
	IX	1086c	115	381	533	373
	X	1073c	115	368	475	352
	XI	1016c	114	315	421	337
	XII	843	103	227	351	310
'30	I	751c	94	165	295	316
	II	759c	92	141	286	340
	III	747c	100	122	298	360
	IV	746c	102	158	337	356
	V	811c	107	192	386	362
	VI	772	91	213	422	357
	VII	792	69	228	412	337
	VIII	896	91	238	412	319
	IX	980	95	245	397	296
	X	962	97	239	352	285
	XI	861	88	195	284	262
	XII	742	78	105	221	224
'31	I	723	83	67	198	230
	II	762	72	58	196	246
	III	813	86	58	199	255
	IV	811	83	91	209	254
	V	855	88	129	272	265
	VI	840	60	145	318	277
	VII	769	54	151	334	261
	VIII	793	58	160	333	231
	IX	826	72	154	310	208
	X	763	79	149	274	190
	XI	659	74	121	236	179
	XII	626	65	74	163	166
'32	I	550	42	46	119	161
	II	630	52	38	124	181
	III	672	50	35	118	192
	IV	681	56	53	152	193
	V	669	67	68	213	202
	VI	678	60	76	241	203
	VII	636	44	78	261	184
	VIII	708	54	81	264	177
	IX	768	53	83	239	162
	X	755	73	73	210	160
	XI	687	68	57	192	166
	XII	560	37	39	145	164
'33	I	494	26	29	116	165
	II	603	39	23	109	187
	III	346b	52	23	124	222
	IV	636	67	35	152	214
	Vd	779	78	49	217	233
	VId	758	71	66	269	247

TABLE II

COEFFICIENTS OF REGRESSION b AND CORRELATION r

Year	Branch of industry									
	Textiles		Apparel		Building		Minerals		Timber	
	b	r	b	r	b	r	b	r	b	r
1927	0,348	0,691	0,613	0,804	1,090	0,952	0,948	0,991	1,025	0,932
1928	0,276	0,257	0,919	0,812	1,095	0,987	0,854	0,990	0,814	0,911
1929	0,721	0,412	0,893	0,895	1,143	0,949	1,206	0,977	0,927	0,958
1930	1,884	0,941	1,611	0,878	1,208	0,960	1,249	0,964	0,958	0,965
1931	0,884	0,567	1,844	0,644	1,464	0,984	1,505[a]	0,914[a]	1,299	0,983
1932	1,673	0,826	3,372	0,876	1,354	0,978	1,817[a]	0,940[a]	1,039	0,920
Normal seasonal indices based on	1925–1931				1925–1928				1925–1931	

[a] If we assume a month's lag after normal seasonal movement the indices become: b 1931—1.596, 1932—1,847; r 1931—0.969, 1932—0.956.

PROGRAM OF THE MEETING OF THE ECONOMETRIC SOCIETY, LEYDEN, SEPTEMBER–OCTOBER, 1933

September 29

10–12:30, 14:30–17:30, 19:30–22:00

R. Frisch: Colloquium-lectures on the algebra of linear transformations and quadratic forms.

September 30

Chairman: H. Schultz, Chicago

10:00—J. Tinbergen: Opening of the meeting.

F. Divisia: A propos du duopole.

O. Lange: On the relation of particular to general equilibrium.

J. Wiśniewski: Interdependence of cyclical and seasonal variations.

Chairman: L. V. Furlan, Basel

14:30—M. H. Belz and F. V. Waugh: Further measurements of money utility.

J. R. Hicks: The application of mathematical methods to the theory of risk.

E. C. van Dorp: On the theory of interest.

Chairman: O. Lange, Cracow

19:30—Colloquium.

October 1

Chairman: R. D'Addario, Roma

10:00—J. Tinbergen: Is the theory of harmonic oscillations useful in the study of business cycles?

Chairman: E. H. Phelps Brown, Oxford

14:30—V. Furlan: Quelques considérations sur les ensembles renouvelés.

J. Marschak: Theoretical problems suggested by Roosevelt's policy.

R. Frisch: Some problems in economic macrodynamics.

Chairman: F. Zeuthen, Copenhagen

19:30—Colloquium.

OCTOBER 2

Chairman: W. J. DE LANGEN, Wassenaar, Holland

10:00—E. H. PHELPS BROWN: Report of the Committee on Source Material for Quantitative Production Studies.

H. SCHULTZ: New investigations in the demand for related goods.

F. ZEUTHEN: On public price policy.

Chairman: M. H. BELZ, Melbourne

14:30—M. KALECKI: Essai d'une théorie des mouvements cycliques construite à l'aide de la mathématique supérieure.

R. D'ADDARIO: Les réserves des sinistrés dans les assurances.

Chairman: F. MACHLUP, Vienna

19:30—Colloquium.

THE MEETING OF THE ECONOMETRIC SOCIETY IN LEYDEN, SEPTEMBER–OCTOBER, 1933

THE third European meeting of the Econometric Society was held in Leyden, Holland, from September 30 to October 2, 1933. The following persons attended: R. D'Addario, Rome; R. G. D. Allen, London; M. H. Belz, Melbourne; H. Bolza, Würzburg; E. H. Phelps Brown, Oxford; B. A. Chait, Antwerp; F. Divisia, Paris; Miss E. C. van Dorp, Bloemendaal; R. Frisch, Oslo; L. V. Furlan, Basle; R. van Genechten, Utrecht; L. Hamburger, Scheveningen; J. R. Hicks, London; M. V. Holtrop, Ijmuiden; N. Kaldor, London; M. Kalecki, Warsaw; J. G. Koopmans, The Hague; O. Lange, Cracow; W. J. de Langen, Wassenaar; A. Lerner, London; G. Lutfalla, Paris; F. Machlup, Vienna; J. Marschak, Oxford; R. Opie, Oxford; S. Posthuma, Bloemendaal; H. Schultz, Chicago; J. Tinbergen, Schevenigen; Miss U. K. Webb, London; J. Wiśniewski, Warsaw; F. Zeuthen, Copenhagen.

A series of "colloquium-lectures" by Professor Ragnar Frish of Oslo, on the algebra of linear transformations and quadratic forms, was given on September 29, the day before the opening of the meeting.

A further "colloquium-lecture" on harmonic oscillations by Professor Ehrenfest of Leyden could not be given as planned, owing to his sudden death. In his opening speech, Professor J. Tinbergen of the Rotterdam School of Economics paid a tribute of personal gratitude to his old teacher, Professor Ehrenfest, and voiced the deep regret of the Society.

The working time of the congress was divided between meetings and informal colloquiums. Most of the papers presented centered around one of the two main topics, I. Business Cycles and II. Theory of Markets. In the following report, the papers and discussions are arranged according to this division; a third section for "Other Subjects" is added.

I. BUSINESS CYCLES

Professor J. Tinbergen of Rotterdam put the question: "Est-ce que la théorie des oscillations harmoniques peut être utile à l'étude des cycles économiques?" He distinguished between the "data" $a, b, \cdots$ —"those quantities which we do not want to explain"—and the variables proper $x, y, \cdots$. If the data are given functions of time, and if all the variables entering into the equations of the system refer to the same point of time, i.e., if the equations are of the form $f(x_t, y_t, \cdots, a_t, b_t, \cdots) = 0$, then the movement is an "exogenous" one, as distinct from the "endogenous" movement of a system with equations of the form $f(x_t, y_t, \cdots, x_{t-\theta}, \cdots, a, b, \cdots) = 0$. In his Lausanne paper of

1931,[1] Tinbergen, starting from an economic assumption such as $x_t = ap_{t-\theta}$ (x being the quantity offered, $p_{t-\theta}$ the price at a past date), arrived at an "endogenous" differential equation yielding in certain circumstances a periodical solution. In Leyden he used a more indirect way of approaching cyclical endogenous movements by starting from the mathematical nature of harmonic oscillations and seeking among the main economic relations those likely to fit into the harmonic pattern, i.e., into a differential equation of second degree with suitable coefficients. Accordingly, he marshalled economic relations into two groups: (1) "differential" phenomena, i.e., mainly functions of the rate of price change $p'(t)$, and (2) "integral" phenomena, mainly functions of $\int p dt$, where p is the price (seasonal and trend influences being eliminated).

To group (1) belong the following phenomena: (a) profits and losses on commodity stocks and investments, and supply depending on such profits and losses, (b) speculative profits and losses and the demand for luxuries depending thereon, (c) relation between the production of consumers' goods and production of producers' goods, and (d) constant short lags T permitting of constructing approximate equations of the type $p(t-T) = p(t) - Tp'(t)$. To group (2) these phenomena belong: (a) the rate of current investments being a function of price (as seems to be empirically demonstrated), the total amount of existing fixed capital ("capacity") would be an integral of this function; (b) commodity stocks are an integral of the surplus of production over consumption while both of them are functions of prices; (c) the annual investment of short credits is the difference between the annual revenue and the sum of consumptive expenses and long investments; the latter fluctuate more violently than the other components, and depend on prices; therefore, the amount of short credits outstanding (excluding "created" credits) is likely to be a function of the price integral; (d) it seems that the rate of wage-changes depends on the degree of employment and therefore approximately on prices; if so, wages would vary with the price-integral, and so would the supply and demand for commodities produced or consumed by wage-earners. Tinbergen's statistical tests had, however, induced him not to give too much credit to most of the hypotheses of group (2), the correlations he had hitherto found being too small.

In the discussion of Tinbergen's paper, B. A. Chait of Antwerp called attention to the fact that profits, and not prices, determine the entrepreneur's decisions. M. Kalecki of Warsaw seconded this contention; he also disagreed with Tinbergen's assigning a relatively low

[1] ECONOMETRICA, I, 36–51.

importance to the fluctuations of the "capacity," i.e., the total amount of existing fixed capital (group 2a); in fact, profit rates and fluctuations of fixed capital are essential for Kalecki's own theory (see below). Tinbergen based his reply on empirical observations, which show a high correlation between profit rate and price level, and which, on the other hand, allow little importance to fluctuations of the "capacity" in the cotton and iron industries. It is, however, different with shipping. It may be noted that Tinbergen's previous work on "Schiffbauzyklus" was based on the assumption of a functional relation between freight rates (and therefore shipping profits and ship-building orders) and total tonnage. On the contrary, in Kalecki's paper presented in Leyden, the price of the product is not regarded as a link between capacity and profits, this necessary link being supplied by some other assumptions (see below). Henry Schultz of Chicago pointed out that the mere process of readjustment of the distorted demand-supply-equilibrium of an individual commodity may be of oscillatory character. Thus, oscillatory functions for individual commodities have been obtained by Evans and Theiss on the assumption that the quantity supplied (as distinct from that demanded) depends on the price existing at the moment of beginning production (as distinct from the moment of selling). But, asked Schultz, why should all the individual oscillations be synchronized and not neutralize one another? He expected that an answer might be found by investigating Evans's equations for important constructional goods. François Divisia of Paris confessed to being rather sceptical about the interpolation of economic time series by harmonic oscillations, either damped or explosive. Ragnar Frisch of Oslo replied that periodic functions, or other kinds of "time-shapes," determined by an economical theoretical set-up, may be regarded as one of two parts—equally essential—in the explanation of observed series; the other part consists in the study of the effects produced when periodic functions are disturbed by erratic shocks.

An attempt towards such an explanation was the main subject of the next paper, "Some Problems in Economic Macrodynamics," by Professor Ragnar Frisch of Oslo. He emphasized that the properties "static" and "dynamic" do not relate to phenomena (which can be either stationary or shifting) but to theories; his definition of "dynamics" corresponded substantially with Tinbergen's definition of "exogenous movements" given above, the problem being to find the "time shape" (length of the cycle, lags, amplitude, etc.) of such movements. Microdynamics are concerned with particular markets, enterprises, etc., while macrodynamics relate to the economic system as a whole. The goods move between the following spheres: "human

machine," land, stock of capital goods (Z), production of capital goods per time unit (y), production of consumers' goods per time unit (x), stock of consumers' goods (X). The rate of increase of capital stock per time unit is, then, $\dot{Z}=y-(hx+ky)$, h and k being technological constants, expressing the wearing-out of capital. While in a stationary society $\dot{Z}=0$ and $y=h/1-k\cdot x=mx$, ($m=$"coefficient of total wearing-out"), in a non-stationary society

$$y = mx + \mu\dot{x}, \tag{1.1}$$

where $\mu=$"constant of fabrication"). This "production policy equation" (1.1) has been suggested by Frisch (1932) as against J. M. Clark, Hansen, and Mitchell, who neglected the term mx.[2] A second equation, called the "consumption equation"

$$\dot{x} = c - \lambda(rx + sy) \tag{1.2}$$

contains c, the "force increasing consumption" and λ, the "brake put on from the monetary side"; r and s are the reciprocal money velocities in the streams of consumers' and producers' goods respectively, so that $rx+sy$ is the Walrasian "encaisse desirée." The oscillatory character of the movement is due to a third relation expressing the lag of decision. In the study of capital production, a distinction was made between capital production starting y and the carry-on activity z. If at the moment t, the new investment orders per time unit and the carry-on activity are y_t and z_t respectively, and if D_τ is a technologically given "advancement function" expressing the progress of construction in time, then

$$z_t = \int_{t=0}^{\tau} D_\tau y_{t-\tau} dt. \tag{1.3}$$

When the distinction between y and z is made, z has to replace y in (1.2).

If construction goes on uniformly, i.e., if D_τ is a straight line, $D=1/\epsilon$, where ϵ is the total construction period and (1.3) reduces to

$$\epsilon z_t = y_t - y_{t-\epsilon} \tag{1.3a}$$

(cf. above, Tinbergen's paper, group 1(d)). The system of (1.1), (1.2), and (1.3a) is solved by putting

$$x = a_0 + \sum_k a_k e^{\rho_k t}, \qquad y = b_0 + \sum_k b_k e^{\rho_k t}, \qquad z = c_0 + \sum_k c_k e^{\rho_k t} \tag{1.4}$$

[2] See discussion in *The Journal of Political Economy*, (Chicago), 1932.

where the ρ's are the roots of the characteristic equation

$$\frac{\epsilon\rho}{1-e^{-\epsilon\rho}} = -\lambda s\frac{\lambda r+\mu\rho}{\lambda r+\rho}. \tag{1.5}$$

Frisch obtained, in his first attempt, the following constants: $\epsilon=6$ years, corresponding to an average construction period of 3 years, $\mu=10$ (a constant without much influence), $m=20$ per cent, $\lambda=5$ per cent, $r=2$, $s=1$. The only real root of (1.5), viz. $\rho_0=-0.0805$, gives an exponential trend with negative exponent. The rest of the roots are conjugate imaginaries and mean cycles, possibly lagging behind one another. The three main periods found by the solution of the characteristic equation were $\rho_1=8.56$ years, $\rho_2=3.49$ years, $\rho_3=2.2$ years. It will be seen that the first two of these correspond almost exactly to the well known long and short business cycles observed in economic data. Professor Frisch said that he believed this correspondence between theory and observation was not wholly due to coincidence in the figures chosen in the numerical example. He warned, however, against attributing too much significance to the numerical results obtained, until further investigations had been made.

Professor Frisch then proceeded to what he considered the most important part of his analysis. The observed time series have an irregular shape and possess neither constant periods nor any systematic tendency towards damping. How can these facts be brought into harmony with the theoretical truth? An analogy was given by a pendulum exposed to erratic blows, which would make the velocity jump discontinuously. If the pendulum started at the initial point t_0, and if there were no blows, the ordinate at the moment t would be

$$y(t) = P(t-t_0)\cdot y_{t_0} + Q(t-t_0)\cdot \dot{y}_{t_0},$$

P and Q being functions of the variable indicated. Each erratic shock e_k applied in the moment t_k could be looked upon as starting a new pendulum; the *sum* of the ordinates of all these pendulums, namely, the moving sum

$$y(t) = \sum_k Q(t-t_k)\cdot e_k, \tag{1.6}$$

will be the total ordinate of the disturbed pendulum. It was previously shown by Frisch that the moving averages of a series of non-autocorrelated numbers form cycles whose periods are $3.5l$, $0.7l$, etc., l being the length of the moving period used. A similar analysis may be carried through if the moving average is a weighted one as in (1.6). The nature of the weights will determine the time shape of the resulting series. In his laboratory in the Economic Institute in Oslo, Frisch had

tested this theory on constructed numerical examples, where the erratic series of shocks was obtained from the list of Norwegian lottery drawings.

The discussion of Frisch's paper related mainly to the economic meaning of his fundamental equations (1.1), (1.2), and (1.3). In reply to Fritz Machlup of Vienna, Frisch stated that these equations do not (or do not necessarily) involve technical progress, but he said that the equations do involve fluctuations in the degree of employment of men and instruments, as well as fluctuations in the volume of purchasing power.

To questions by T. G. Koopmans of The Hague and M. Kalecki of Warsaw, Frisch answered that λ is a constant expressing the intensity of the "brake" effect coming from the monetary side, and the constant c is the expression of the "normal" magnitude of the second term to the right in (1.2), the notion of the "normal" being defined by $\dot{x}=0$. F. Divisia of Paris, suggested that the exogenous erratic shocks be interpreted as "écarts" of human behavior from the routine supposed by Frisch's equations. Answering Divisia's question as to which are the parameters in these equations corresponding to the acceleration term in the ordinary harmonic equation, Frisch suggested that there is an analogy between the acceleration and the right-hand part of his equation (1.3). He emphasized, however, that the ordinary harmonic equation is too simple a pattern in the present case. Henry Schultz of Chicago put the general question as to whether definite "principles of conservation" analogous to those of mechanics may be supposed to rule our economic dynamic system. Is it the principle of profit maximization? Would a Soviet economy present a different dynamic system? In his answer Frisch denied that the profit maximization principle could be taken as a general principle governing the economic system. We see, for instance, that in a depression the entrepreneurs double their efforts to minimize cost through rationalization and otherwise. If such cost-cutting is possible, why did they not do it before? There seems to be present a phenomenon of inertia—or perhaps friction—that is in opposition to a general principle of profit maximization. This point had been emphasized by Professor Fréchet in a recent discussion at the Institut Henri Poincaré.

M. Kalecki (Institute of Economic Cycles and Price Research, Warsaw) delivered an "Essai d'une théorie des mouvements cycliques." He assumed that the ratio, at any moment t, of the orders I for instrumental goods per time unit, to the existing volume of such goods, K, say $I(t)/K(t)$, is an increasing function of the rate of net profit prevailing at the moment t. He defined net profit as the difference between the gross profit (i.e. the sum of capitalists' consumption and investment,

including replacement, and neglecting the changes of inventories), and the sum of replacement costs and interests. The gross profit is an increasing function of the investments A, and its ratio to the value of existing capital K is an increasing function of A/K, or of $(A+B_0)/K$, B_0 being a constant expressing that part of capitalists' consumption which does not change when profits change. Since, on the other hand, the rate of replacement is supposed by Kalecki (contrary to Frisch, cf. above) fairly constant, it follows that

$$\frac{I}{K} = F\left(\frac{A + B_0}{K}, p\right); \qquad \frac{\partial F}{\partial\left(\frac{A + B_0}{K}\right)} > 0; \qquad \frac{\partial F}{\partial p} < 0,$$

where p is the interest rate. Although in Kalecki's opinion p is likely to increase when $(A+B_0)/K$ increases, he assumed I/K to be on balance an increasing function of $(A+B_0)/K$ alone. If linear relations be supposed, we ought therefore to have

$$I = m(B_0 + A) - nK, \tag{2.1}$$

where I, A, K, are functions of t, and B_0, m, n, are, according to Kalecki, positive constants. If the average duration of the construction period of finished instrumental goods is ϑ, and consequently the shipments of these goods at the moment t are $I(t-\vartheta)$, and if the average investment orders per time unit, supposed to be, in an "economy without trend," equal to the replacement costs, are I_0, then

$$\frac{dA}{dt}\cdot\vartheta = I(t) - I(t - \vartheta), \tag{2.2}$$

and

$$\frac{dK}{dt} = I(t - \vartheta) - I_0, \tag{2.3}$$

therefore, from (2.1)

$$\frac{dI}{dt} = \frac{m}{\vartheta}[I(t) - I(t - \vartheta)] - n[I(t - \vartheta) - I_0], \quad (m > 0, n > 0). \tag{2.4}$$

This differential equation resembles the equation

$$f'(t) = -af(t - \vartheta), \qquad (a > 0), \tag{2.5}$$

proposed in Tinbergen's "Schiffbauzyklus" (*Weltw. Archiv.* July, 1931). Following Tinbergen's method, Kalecki finds one periodical solution

$$T = T_0 + De^{(m-x)t/\vartheta} \sin y^{t/\vartheta},$$

D being an arbitrary constant, and x and y depending on m, n, and ϑ. The cycle period is $T=2\pi/y$. Assuming the case of constant amplitudes, so that $x=m$, we have from (2.1) and (2.4) the relations permitting one to find m, n, y, and T, if ϑ, I_0/K_0, and B_0/K_0 (K_0 being the average existing volume of instrumental goods), are known. If $\vartheta=0.6$ years, the average amortization rate $=0.05$ (a parameter without much influence), and $B_0/K_0=0.13$, the period T obtained turns out to be 10 years.

It was suggested in the discussion that the important divergence in the numerical estimate of the duration of construction (0.6 years in Kalecki's paper but 3 years in Frisch's) ought to be the subject of a careful statistical investigation. On the other hand, the duration of destruction does not influence Kalecki's result very much; this is due, according to a remark of Tinbergen's, to the assumption Kalecki has made as to the replacement orders being constant. Replying to a question by B. A. Chait, Kalecki emphasized that only the period, but not the amplitude, of the cycle can be deduced from his fundamental equation (2.4). R. Frisch doubted that there was only one solution of (2.4) yielding $T>\vartheta$, a property which Kalecki claims for this equation in analogy to Tinbergen's equation (2.5).[3] Kalecki agreed with Fritz Machlup that a credit inflation was necessarily implied in his explanation of the business cycle; the additional amount of credit was, however, not necessarily equal to the amount of imposed saving.

The paper presented by J. R. Hicks of the London School of Economics on "The Application of Mathematical Methods to the Theory of Risk" may also be placed in the group of papers dealing with business cycles. The situation considered in this paper was primarily that of the investor, who is considering the investment of a given quantity of resources in one or the other of a number of possible investments, in all of which the yield is uncertain. It was suggested that, with certain qualifications, the "prospect" of each investment might be represented by a frequency curve, giving the probabilities of yields of various amounts, as they appear to the investor in question. The shape of this curve will depend on (a) his judgment, and on (b) his

[3] The uniqueness proof presented by Kalecki in his lecture did not seem convincing because his characteristic equation was not *identical* with Tinbergen's equation (for which the existence of a unique solution in the range considered is proved), but was a *transformation* of Tinbergen's equation, and Kalecki did not discuss how the ranges of the solutions behave under the transformation. Later I deduced Kalecki's result by an independent method not using Tinbergen's equations. This seems to show amongst other things that Kalecki's *result* is actually correct: there exists only one solution of the kind considered by Kalecki.—EDITOR.

preference for one set of chances as against another set. It was this second characteristic which was studied.

"The various frequency-curves of different investments could be arranged in an order of preference, which could be approached in the same way as the preference-scale used by modern writers in the theory of wants. In order to investigate the nature of this scale, the shape of each frequency curve may be studied by means of its *moments*—in the statistical sense. Each curve could be rigidly defined by taking a sufficiently large number of moments, and an approximation to the situation obtained by taking a small number."

The case where two moments (the mean value and the standard deviation) only were considered, was next discussed in detail. J. B. Clark's doctrine that "business risks are carried for their subjective actuarial value" was examined in the light of this and rejected: "(a) because it failed to allow for the profitability of creating certain sorts of risks, and (b) because it assumed the measurability of utility, in the sense shown by Pareto (*Manuel*, p. 541) to be impossible. The "risk-indifference-map" must be regarded as an independent *datum* of economic analysis, on a par with other 'tastes and obstacles.'"

In a final section, this theoretical framework was applied to the theory of money. "The holding of money being an "investment" like any other, it follows that changes in the risk-indifference-map, and the impact of changes in other data on the risk-indifference-map, must be among the main factors affecting the demand for money; so that they must be introduced as a very significant element into the theory of trace fluctuations. It appears to follow from this line of approach that the decline of "Savings" during a trade depression is not so certainly favorable to recovery as has sometimes been supposed."

"The Correlation Between Seasonal and Cyclical Variations" was discussed by J. Wiśniewski of the Business Cycle and Price Research Institute of Warsaw. He applied the method of Simon Kuznets (*Journal of the American Statistical Associati[illegible]*, 1932) for measuring (1) the degree of similarity r_{sd} between the standard seasonal pattern and the observed seasonal pattern, and (2) the regression coefficients b_{ds} of amplitude of the seasonal movement. b_{ds} proved to be, in a number of Polish and German seasonal industries, a decreasing function of the level of employment. The economic explanation is given in the fact that, during a slump, the production is concentrated in the best weeks, whereas in a boom those weeks which previously were "submarginal" prove also to be worth while to work in.[4] Henry Phelps Brown of

[4] A different result seems, however, to have been obtained in the recently published "Ausschaltung von Saisonschwankungen mittels Lag-Korrelation,"

Oxford called attention to the influence which the moving average (as used in Kuznets's method) must exert upon the result in affecting the apparent degree of seasonal variation where a change of trend occurs.

Two further special questions relating to business cycle policy were the subject of a communication of "Theoretical Problems Suggested by Roosevelt's Policy," by J. Marschak of Oxford. This policy may be supposed to have three aims: (1) to secure by a higher general price-level a redistribution of real debts and of some current incomes; (2) to encourage by windfall profits, during the transition period of rising prices, entrepreneurs' activities; (3) to reach, with positive but not necessarily high profit rates, full utilization of the capacities erected in the previous period of technical progress, by supplying purchasing power MV to non-hoarders.

Analogous principles applied to situations like 1923–29 (rapid technical progress) would mean inflation of wages, in order to prevent either falling prices (which lead to shifting of debts, windfall losses, and subsequent hoarding), or profit inflation, which is unable to persist without steadily growing volumes of credit. These principles cannot, however, secure full employment without further control of produced quantities, since it is impossible to keep simultaneously a desired number e of employed productive factors and a desired price-level P of consumers' goods. If q_i is the quantity produced and consumed, and p_i the price of the commodity i ($i=1, \cdots, n$), and if w is the remuneration per unit of productive factor, λ the function used to calculate the price-level P, and σ_i and δ_i the supply and demand functions respectively, then

$$q_i = \sigma_i\left(\frac{p_i}{w}\right) = \delta_i(p_1, p_2 \cdots w \cdot e),$$

$$\sum p_i q_i = w \cdot e = MV,$$

$$P = \lambda(p_1 \cdots p_n, q_1 \cdots q_n),$$

MV, p_i, q_i, and w, being the $(2n+2)$ unknowns in $(2n+3)$ equations. This system is consistent only if the functions σ_i and δ_i have definite parameters, which is by no means guaranteed by the historically given technique and taste unless there is artificial control. The control of goods with very inelastic demand (mainly agricultural products), or with very elastic supply, is a practical instance of deliberately influencing σ_i and δ_i.

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by Franz J. Zrzavy (Wien, 1933), Verlag des Oester. Inst. für Konjunktionsforschung. The paper of J. Wiśniewski is contained in *Prace Instytutu Badania Konunjktur Gospodarozych i cen*, Vol. II, No. 2, Warsaw, 1933.

This point was criticized by F. Divisia, who denied the practical possibility of controlling even the quantity (MV), let alone the direction of the purchasing power injected.

Marschak treated further the problem of secondary unemployment, or, generally, of the propagation of equilibrium-distorting through the economic system. The system in equilibrium may be expressed by a matrix $\boldsymbol{A}$ whose element a_{ij} is the value of the goods bought per time unit by the individual (or group) i from the individual (or group) j. The equilibrium is characterized by the fact that $\sum_i a_{ij} = \sum_i a_{ji}$. If an individual, with total receipts r_h, is cancelled out, two extreme cases I and II are conceivable. The element a_{hk} may be $= r_h$ or $= 0$ according to whether $k =$ or $\neq h$ (Case I, a hermit killed by lightning), or according to whether $k =$ or $\neq (h+p)$, where $p = 1$ or an integer having no common divisors with N when $N =$ number of individuals (Case II, treated by Pigou and Kahn). In the new equilibrium $\boldsymbol{A}'$ we must have in Case I, the element $a_{ij}' = 0$ if either i or $j = h$, otherwise $a_{ij} = a_{ij}'$; in Case II, always $a_{ij}' = 0$. The total shrinkage is $q = r_k$ in Case I, and $q = N \cdot r_k$ in Case II. Special intermediate cases may also be constructed, e.g., $a_{hk} = a_{kh}$, or $a_{hk} = 0$, according to whether $k =$ or $\neq h, i, j$; then $q = a_{hh} + a_{hi} + a_{hj}$. A way to the general solution was suggested by Ragnar Frisch in an informal colloquium. The solution as the present writer sees it may be stated as follows: Given each single expenditure item a_{ik} of each individual i as a function $f_{ik}(r_i)$ of its total expenditures r_i, the equilibrium condition is $r_i = \sum_k f_{ik}(r_i) = \sum_k f_{ki}(r_k)$. If linear functions be assumed, $f_{ik}(r_i) = b_{ik} r_i + c_{ik}$, where necessarily $\sum_k b_{ik} = 1$ (3.1) and $\sum_k c_{ik} = 0$ (3.2). Let $s_i = \sum_k c_{ki}$, $b_{ik}^* = b_{ki}$. The equilibrium system (denoting vectors and matrices by heavy-faced letters), $(\boldsymbol{B}^* - \boldsymbol{E})\boldsymbol{r} = -\boldsymbol{s}$, has an infinite number of solutions in $\boldsymbol{r}$, all lying on a straight line, since, by (3.1), $|\boldsymbol{B}^* - \boldsymbol{E}| = 0$, and, by (3.2), $\sum s_i = 0$. Only those points $\boldsymbol{r}_\alpha \cdots \boldsymbol{r}_\gamma \cdots$ on this line, which correspond to integral numbers of indivisible producing and consuming units, and which do not exceed certain limits given by the productive capacity of the country, have an economic meaning. If an equilibrium $\boldsymbol{A}_\alpha$ corresponding to a solution $\boldsymbol{r}_\alpha$ is distorted by new taste and technique, i.e., if for each old single expenditure function f_{ik} a new function f_{ik}' is substituted, the new equilibrium $\boldsymbol{A}_\alpha'$ and the new total expenditures $\boldsymbol{r}_\alpha'$, which may be economically expected, are presumably such that for the most values of γ the modulus $(r_\alpha' - r_\alpha) \leqq (r'_\gamma - r_\alpha)$. To F. Divisia's remark that in the long run the unemployed factors (as stated by J. B. Say) must disappear, Marschak replied that the adaptation may in practice take a long time, and that, even in the "long run," the condition of employing all given resources may imply negative prices or incomes, and therefore be economically impossible.

II. THEORY OF MARKETS

In a paper, "A propos du duopole. Fluidité d'un marché et élasticité d'une clientèle," Professor François Divisia of the École Polytechnique, Paris, showed that the classical theory of duopoly, as proposed by Cournot, attacked by Edgworth and Pareto, and restated by more recent writers, needs a more general and realistic formulation of its assumptions. It is not necessary to assume a rigid borderline between the cases of acquiring and non-acquiring new buyers; further, it is not necessary to assume that each duopolist supposed the production of the other to be constant. The concrete expectations of each duopolist as to the behavior of the other, their way of regarding each other *en chiens de faïence* is to be studied statistically. Two simple instances of statistical study were given: (1) a gas and an electricity plant supplying quantities d_1 and d_2 (Amoroso), and (2) a port and a railway line running to it, selling their respective ("completing") services for the prices p_1 and p_2. The solution depends on the total demand function $f(D, P) = 0$, and on either (in Case (1)) $d_1+d_2=D$, or (in Case (2)) $p_1+p_2 = P$. A further generalization is required by the fact of unequal prices due to the imperfect "fluidity of the market." This latter is measured by $dQ/d(P-p)$, where P and p are the respective prices charged by the duopolists, and Q the quantity sold by one of them; the "élasticité de la clientèle" is

$$\frac{PdQ}{Qdp} = \frac{P}{Q}\left(\frac{\partial Q}{\partial p} + \frac{\partial Q}{\partial (P-p)}\right),$$

where $Q=Q(P, P-p)$. More generally

$$Q = Q(P, \phi(P, p)). \tag{4.1}$$

In the discussion, F. Zeuthen of Copenhagen showed "that even more than two prices are possible within a duopolistic market, according to the local distance, the "distance" in the quality or the "distance" with regard to the personal preferences. (Cf. Hotelling in *Economic Journal*, 1929, and Zeuthen in *Quarterly Journal of Economics* and in *Nationalokonomisk Tidsskrift*, 1933). In a market with a number of sellers and a number of buyers spread over a surface (three or more dimensional if quality and personal connections are taken into consideration), it is possible to find a definite equilibrium with varying prices and a semi-monopolistic policy for each seller. As to Professor Divisia's "élasticité de la clientèle," we have to distinguish between the capacity to take customers from competitors and the capacity to gain fresh consumers by price reductions." Instead of a definite difference in price, Zeuthen rather believed in different "powers of exten-

sion," or different elasticities. If these and the aggregate demand curves are given, it is possible to find a definite solution. "If any seller, by first making a reduction in price, is able to extend his sales in accordance with the common demand curve of the market, but is unable to take customers from the other sellers or to keep the customers thus taken if the other sellers follow suit in the price reduction, we have Cournot's solution. If a minimal price reduction takes all customers from the competitors until they themselves reduce their price still more, price is reduced to costs. Most probably, we get a price between those possibilities, and different prices sold for each seller determined by his power of extension and cost conditions (cf. Chapter II of Zeuthen's *Problems of Monopoly*).

"If the conditions are not static, the chronology and tactics are of importance, i.e., it is important which of the parties takes the first step, and what one party believes about the probable behavior of the other party (cf. Frisch's "conjectural elasticity"). Movements here are slow and time is sufficient for a static equilibrium to be reached. Also, the price policy is complicated because a series of future expected conditions are considered. The field is open for cut-throat competition, bluff, education of customers by low prices, or exploitation of passing opportunities by high prices, and all kinds of personal and institutional influences."

R. Frisch replied to Zeuthen that the identical phenomena expressed by the conjectural elasticities, or the conduct *en chiens de faïence*, or Divisia's equation (4.1), are not confined to dynamics. When Schultz suggested the expression of the "conjectural" relations indifferently either as $F(P, p) = 0$, $f(P, p) = 0$, or as $G(Q, q) = 0$, $g(Q, q) = 0$, Frisch called attention to the fact that such a substitution of quantities for prices as the independent variables is not possible if the elasticity of demand is very small. Kalecki showed that the incomplete utilization of the capacity of enterprises, though impossible in a perfectly fluid market with equal prices, is a fact which proves the generalization made by Divisia to be important. A discussion on this point, opened by E. van Dorp of Bloemendaal, Holland, and N. Kaldor of the London School of Economics, was closed by Tinbergen's and Divisia's remarks as to the factual existence of price inequalities requiring explanation and empirical study.

Professor Zeuthen started in his paper on "Public Price Policy" with the analysis of a public enterprise desirous of maximizing the sum of its own profit and consumers' rent, the latter reduced by a factor from 1 to zero. Contrary to Marshall, he considered not the whole of the demand curve, but only alternative points and a special kind of elasticity. "In the case of small variations, the elasticity of

demand price in excess of operating costs $e=(r-1):r$, r being the relation between money increase in consumers' rent and decrease in the profit of the State as a consequence of a price reduction; e is supposed to be constant, as it very nearly is in most practical cases. The greater the interval between alternative prices, the greater the change in consumer's rent compared with the change in profit. Consumers' rent may be measured in money if changes affect only a small part of income; otherwise arbitrary modifications are necessary, according to difference in income levels and assumed utility conditions. Dealing with several uses of the same public utility, the different combinations between (1) free and scarce capacity and (2) consumer's rent, (a) valued equal to, (b) valued less than, profit and (c) completely disregarded, were considered. Where the profit was valued higher, we got a price differentiation between the groups in accordance with their excess elasticities and special complications where also capacity was scarce. By seasonal or daily variations approximately full use of capacity at any moment is the ideal, except where the taxation point of view requires a higher price. For instance, in the Copenhagen Telephone Company a variation of payment in accordance with the regularity of the different uses of capacity is applied based on calculations of probability. A rational consideration of consumers' rent at the time of consumption and price fixing requires a careful restriction of capacity at the time of building. Here, an extensive knowledge of demand in similar fields is required." As to the possible political motivations, Zeuthen's article in the *Economic Essays* offered to Cassel was referred to.

Oscar Lange of the University of Cracow, Poland, analyzed the "Relation of Particular to General Equilibria." If a denotes a small positive number, we may disregard in each of the Walrasian demand or supply functions for the commodity i, say, $F_i(p_1 p_m)$, $S_i(p_1 p_m)$, all those variables j in respect to which the absolute value of the partial elasticity $|\eta_{ij}| < a$. The reduction of the number of variables to one is, however, impossible in the case of joint and composite demand and supply. On the other hand, frictions due to the lack of divisibility make some partial elasticities very small and the procedure a realistic one. In the discussion, the procedure proposed by Lange was criticized by A. P. Lerner of London, instancing the case where the sum of partial elasticities exceeds the direct elasticity; and by J. G. Koopmans of The Hague in another case, where the price of some other commodity affected the quantity bought of the commodity considered more than its own price. Schultz stated, however, that the probability of the first case was not great owing to differences in sign between the various partial elasticities. As to the second case, Lange claimed to have already dealt with it under the head of "joint and composite demand and supply."

The problem of interrelated demand and supply was more closely treated by Professor Henry Schultz of the University of Chicago, who presented the results of "New Investigations in the Demand for Related Goods," adding some new points to his earlier results (*Journal of Political Economy*, Aug. 1933). If x_i and x_j are the quantities, and y_i and y_j the prices, of two commodities, these are said to be completing, independent or competing according as

$$\frac{\partial^2 \phi}{\partial x_i \partial x_j} \equiv k \frac{\partial y_i}{\partial x_j} \gtreqless 0, \tag{5.1}$$

k is here the final degree of money utility and assumed to be constant, ϕ is the total utility function $\phi(x_1 \cdots x_n)$ and assumed to be finite and to have continuous second derivatives in the region considered. The last assumption implies, in accordance with an earlier statement of Hotelling's, the integrability condition

$$\frac{\partial y_i}{\partial x_j} = \frac{\partial y_j}{\partial x_i}. \tag{5.2}$$

It follows, in the case of linear equations such as

$$\begin{aligned} x_1 &= h_1 + r_{11}y_1 + r_{12}y_2 \\ x_2 &= h_2 + r_{21}y_1 + r_{22}y_2 \end{aligned} \tag{5.3}$$

that $r_{12} = r_{21} \gtreqless 0$ according as the commodities are completing or competing. M. Ezekiel's data (ECONOMETRICA, April, 1933, page 180), which permit of constructing regression equations connecting prices and quantities of pork (p) and beef (b) with an index of payrolls lagged three months, yield $r_{pb} = 0.0829$, $r_{lp} = 0.0637$. In some other cases, however, the condition (5.2) was not satisfied even with respect to sign. H. Schultz advanced several hypotheses in explanation of this. In the discussion, Phelps Brown asked whether the explanation might lie (1) in the fact that the marginal utility of money is not constant, and (2) in structural changes. A discussion led by R. Frisch on structural and phase relations was continued in an informal colloquium.

M. Belz of Melbourne, Australia, reported on "Further Measurements of Money Utility" done in cooperation with R. Frisch and F. W. Waugh of Boston, Massachusetts, in the Oslo Economic Institute. The theoretical scope, viz., the finding of "consumption surfaces" $f(r, \alpha, x) = 0$ for single commodities (where r = real income, x = quantity consumed, α = inverse of the real price), and the calculation of the money flexibility

$$-\breve{w} = \frac{d \log \alpha}{d \log r},$$

was, as has been suggested in previous publications of R. Frisch. Apart from r, α, and x (or their logs), three further variables were introduced: time and the annual increments of log α and log r. The main interest of the paper lay in the devices applied for selecting, out of these six theoretically involved variables, a smaller set of variables which should yet show the closest possible approximation to a linear interdependence of the variables. An observed n-dimensional scatter diagram is the "flatter" (the "scatterance" is the smaller), the more the correlation determinant

$$\begin{vmatrix} 1 & r_{12} & \cdots & r_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ r_{n1} & \cdots & & 1 \end{vmatrix}$$

tends to zero. To test the "scatterance" of different possible sets, various methods were applied. These tests indicate the selection of r, α, and x, and give approximately identical results for the money flexibility when butter and coffee (U.S.A. 1909/31 and 1913/31) were used as comparison commodities, viz. $-\breve{w}=1.5$. This fairly accords with the results obtained in Frisch's studies based on sugar consumption during the French inflation, but not with the results of his "budget method" applied to U. S. data for 1918/19. In order to apply the "budget method," it was planned to use other budgets than the American of 1919 (which was based on very high incomes). The results obtained by using meat as the reference commodity may be improved by bringing the prices of the meat-substitutes into the set of other variables. A plan for further work in this direction was outlined.

In the discussion, Professor Henry Schultz claimed greater simplicity for the measurement of standard errors; it involved the evaluation of the scatterance coefficient and gave substantially the same results. Professor Ragnar Frisch proposed to test the various methods by applying them to arbitrarily constructed numerical examples with known linear relationships and known erratic elements.

III. OTHER SUBJECTS

Miss E. C. van Dorp of Bloemendaal, Holland, in a paper "On the Theory of Interest" pleaded for collaboration between economic theory and mathematical method. There is no margin left for profit in an equilibrium theory; a wage fund theory, must, therefore, be substituted for the equilibrium theory if the interest rate is to be explained.

Professor Louis Vladimir Furlan of the University of Basle gave in his paper "Sur la théorie des ensembles renouvelés" a further development of F. Divisia's proposals published in ECONOMETRICA (July, 1933). To introduce the aspect of "ensembles renouvelés" into eco-

nomics would imply a reconstruction of the whole of economic theory. The economic system as a whole is an "ensemble renouvelé," whose elements are the individuals, their services and actions; but there are also other "ensembles" subordinated to this. A distinction might be made between "des ensembles qui se renouvellent par eux-même" (e.g. population), and all others; this distinction proves, however, to be rather a quantitative than a qualitative one. Even where a complete independence between the present and past debits seems to exist, "bio-economical" functional relations between them, sometimes of a very complicated nature, can be found. Furlan gave as an example a formula for the actuarial value of the total costs of subsistence if the movement of population, the minimum age of earners, the costs of education, and incomes, are known.

A "Peculiarity of Casualty Insurance" was shown in a paper by Professor D'Addario of the University of Rome. Let R be the reserve at the year end, P the annual payments, N the number of accidents. Italian as well as French insurance returns proved that $R/P = \text{constant}$ and $(R+P)/N = \text{constant}$. In the discussion, G. Lutfalla, Ministère du Travail, Paris, expressed doubt of the statistical validity of the private insurance figures on reserves.

In the absence of Professor A. Bilimovic (Ljubljana, Jugoslavia) his paper "Wie können unmessbare psychische Grössen in das Gleichungssystem der Wirtschaftsmechanik eingefügt werden" to appear in *Zeitschrift für Nationalökonomie*, criticizing the conception of indifference lines and pointing out the discontinuous character of economic phenomena, was read in extracto and commented on by R. Frisch. The following papers were read by title: Ewald Schams of Vienna, "Formulierung eines Systems ökonomischer Grössen mit Hilfe des Elastizitatsbegriffs," and a paper by Professor W. Waffenschmidt of Heidelberg, on "Die graphische Methode in der Wirtschaftskinetik."

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THE MEETINGS OF THE ECONOMETRIC SOCIETY IN PHILADELPHIA AND BOSTON, DECEMBER 1933

A MEETING of the Econometric Society was held in Philadelphia, December 27–29, in connection with the annual meetings of the social science societies. Headquarters were maintained at the Benjamin Franklin Hotel, where all sessions were held. All the meetings were very well attended and the meeting places in almost every case proved too small for the numbers who wished to attend. The program was well received and gave rise to interesting discussions. One significant feature was the number of younger men who evinced interest in the objectives of the Econometric Society.

The Philadelphia sessions afforded an opportunity for a meeting of the Council of the Econometric Society, at which various items of business were transacted. The Council reelected all the officers of the Society. Professor Joseph A. Schumpeter was reelected to succeed himself on the Council, and Professor Gustavo Del Vecchio of the University of Bologna was elected to the Council in succession to Professor Luigi Amorso of the University of Rome, who had resigned. The Council nominated for Fellowship in the Society twenty-three persons, on whose election the present body of Fellows will ballot. Further, the Council voted to raise the dues of members not resident in the United States or Canada from $3.00 to $4.00 per annum, the new rate to apply to members who should be elected between the date of the Council's action and September 30, 1934, and after that to all members not resident in the United States or Canada.

A meeting of the Fellows of the Econometric Society was also held at Philadelphia, but the attendance was small and no business was considered.

On Wednesday, December 27, at 10 A.M., the Econometric Society held a joint meeting with the American Economic Association, in the Franklin Room of the Benjamin Franklin Hotel. The general subject treated was "Imperfect Competition." Professor Schumpeter of Harvard University acted as Chairman.

Professor E. H. Chamberlin of Harvard University spoke on "Monopolistic Competition and the Productivity Theory of Distribution." Professor Chamberlin noted that the marginal productivity theory of distribution was a part of the purely competitive structure of economic theory. The recognition of monopoly elements in the price system vitiated its conclusions. Whether under pure competition or under monopolistic competition, each factor always received its marginal value product to the individual firm. But, Professor Chamberlin said, whereas under pure competition this was equal to its marginal product

both to the firm and to society, under monopolistic competition it was inevitably less than its marginal product to the firm and bore a very uncertain relation, if any, to its marginal product to society. Furthermore, selling costs being wholly incompatible with the theory of pure competition, the marginal productivity theory in its conventional form indicated for them a productivity of zero.

The second speaker, Professor William Jaffé of Northwestern University, treated the subject "Imperfect Competition—A Problem in General Equilibrium." It was the contention of Professor Jaffé's paper that the current solutions of monopoly or oligopoly prices yielded results that were descriptive of partial equilibria, or at best of temporary equilibria. Definitive equilibria could only be attained when monopoly or oligopoly prices were determined within a system of general equilibrium such as the Walrasian total system. By a suitable adaptation of the Walrasian system, which was originally established for the pure competitive case, to conditions of imperfect competition where monopolists' or oligopolists' profits become an item of cost of production, definitive equilibrium prices and quantities were reached. A sample of such a solution was offered. This solution showed that it was not sufficient to study the behavior of sellers alone, or of buyers alone, and that it was indispensable to study the behavior of both simultaneously in their mutual reactions.

The third paper of this symposium was that of Professor Theodore O. Yntema, of the University of Chicago, on "Imperfect Competition: Some Aspects of Cost and Demand." After stating the two conditions for equilibrium in an industry, the assumptions basic to the analysis were outlined by Professor Yntema, and several useful theorems relative to the cost and demand curves for the individual firm were developed. From these the effects of advertising on size of firm and on price were deduced. Under most conditions, the speaker held that the introduction of advertising into an industry characterized by monopolistic competition would enlarge the size of firm. The factors making for a lower price were an increase in elasticity of demand and falling marginal cost of production, while those leading to a higher price were a decrease in elasticity of demand and rising marginal cost of production for the product of the individual firm.

In the discussion which followed, Professor Harold Hotelling of Columbia University said: "The three papers we have just heard illustrate nicely the three methods of setting forth economic ideas, Professor Chamberlin by means of graphs of demand and other curves, Professor Jaffé with formulae, and Professor Yntema by word of mouth. We may in the future look to an increasing interaction and simultaneous use of these methods. The graphic methods are clearest

in many cases, particularly when only one commodity is involved. With two commodities or more, each with variable price and quantity, too many dimensions are necessary, and then formulae are more useful, at any rate if we are to study interactions. To make the formulae intelligible and to get on with the study, a good notation is essential. Much more attention might be given to the principles of good notation than has generally been done in economic writings. That of tensor analysis is in many points of economic theory most useful.

"Much of economic theory," Professor Hotelling continued, "may be regarded as a set of problems in the theory of maxima and minima. If we knew the demand and cost functions, the problems would be of the straightforward types found in calculus books: to find the prices or quantities giving a maximum, and to prove stability by means of the second derivatives. To this would be added the complications resulting from fixed relations among variables, such as that of a limited budget. However, we do not have the data to make the solution in numerical terms of such problems useful. What we can do is to assume that equilibrium does exist, and then proceed to translate the conditions on the second derivatives into conditions on the first derivatives of the demand functions. It is in this way that the important inequalities of economic theory may all be derived. This provides a unifying principle for the three papers presented.

"In the discussion of competition among a listed number of entrepreneurs," Hotelling concluded, "one point which seems to have been missed by the preceding speakers, and indeed generally in the discussion of the matter, is the fact that the different sellers are likely to have, not one price, but as many prices as there are sellers. This condition, which was emphasized in my paper in the *Economic Journal* of 1929, makes a vast difference in the nature of the competitive situation, and leads to interesting results of several kinds. It is a consequence of the existence of groups of customers having preferences for particular sellers, so that each seller's demand function is a continuous function of all the prices."

The afternoon session of December 27 was held in the La Fayette Room of the Benjamin Franklin Hotel, at 2:30 o'clock, when the subject "The Time Element and Economic Theory" was considered.

Professor Griffith C. Evans of Rice Institute opened the symposium with a paper on "Difference Between Lag and Anticipation in Economic Theory."

Dr. Charles F. Roos of the National Recovery Administration continued with a paper on "Studies in Competition and Production Incentives." Theoretical economists have given inadequate attention to the question of how profits, for the individual producer and for the

industry, as quantities depending upon the price of the product and the rate of production, depend also upon the elasticity of demand (percentage increase in sales accompanying a one per cent decrease in price) for the product, Dr. Roos contended. By the use of mathematics, he indicated that the profit curve for a product having sensitive demand is flatter near the peak than one for a product having non-sensitive demand. In the case of a monopolist who had non-sensitive (stable) demand, such as a public utility, it might be desirable to have some public regulation, since the profit curve had a steep peak and the accounting department of the utility had little difficulty in locating the production rate yielding maximum profit. On the other hand, Roos held that at present there need be no fear of an automobile monopoly, since the demand was so sensitive to price changes that the profit curve was relatively flat and the accounting department was unable to locate the true maximum. In the case of the automobile, however, purchasing power was a more important factor than price. In all successful studies of demand it had been necessary to deflate price by some index of purchasing power; in the case of some agricultural products the bureau of Labor Statistics' index of wholesale prices was satisfactory, but in the case of automobiles, wage indices were better. Purchasing power entered to different degrees in importance in the demands for different commodities, but all demands depended upon it. Dr. Roos demonstrated the theorem that, in general, as purchasing power increased, the rate of production yielding maximum profit increased, and the profit curve had decreasing curvature at the peak, i.e., the curve was sharper. Thus, as purchasing power increased, it became relatively easier for producers to determine operations that yielded maximum profit, but the situation was essentially an unstable one. If the situation in some important industry, such as building, resulted in widespread unemployment (3,000,000 to 5,000,000 men), decrease in purchasing power was sufficient to force retrenchments on other industries. As cost adjustments took place, very generally the very process of decreasing production to increase profits would bring about changes in purchasing power that required smaller and smaller rates of production. Thus a shrinkage in purchasing power could bring about an appearance of general overproduction. Operations of manufacturers depended very largely upon expected future prices. In the economic downtrend, expected price decreased and employment in capital goods industries was delayed. At some stage the downtrend in the psychological factor was arrested and expected price increased so that production and sales were speeded up. In this process of reversal, employment in consumer goods industries was increased. The accompanying increases in purchasing power usually brought out new bank loans (newly created

credit) which were essentially additions to purchasing power, and savings were put into use to construct capital goods.

In view of the studies Dr. Roos presented, it appeared to him that economists had generally written the equation of exchange incorrectly. The equation as it was used today applied only to a consumers' goods economy. To apply it to a capital goods economy it must be written to contain stocks and bonds on the monetary side, and capital goods capitalized as expected future income, plus some small fraction of cost, on the goods and services side. When it was rewritten in that form, the manner in which the psychological factor, debts, interest, etc., could affect the economy was made obvious.

Professor Harold Hotelling of Columbia University followed with a paper, "Prosperity through Increased Production." Professor Hotelling said, in part, "The success of the government's recovery program, or of any national economic program, must be judged, not in terms of price levels, but in terms of the quantities of physical goods and services which are put into the hands of consumers. With this is to be considered the effect of the program on the distribution of wealth among different classes. But the chief thing needed is to increase physical production. In this respect much that is being done at Washington is definitely in the wrong direction. The attempts to increase the prices and curtail the production of oil, agricultural products, and other commodities, are anti-social. Instead of decreasing production, the government might well take measures to increase production, by intervening to lower certain important prices which have not fallen as much as the general level during the depression, and in some cases have actually risen. Most notable among these prices, as barriers to prosperity, are railroad and utility rates. It is suggested that the government operate the railroad and utilities, charging far lower prices for their services than now prevail, and paying overhead costs by means of income, inheritance, and land taxes. This would bring about a tremendous increase in traffic and in business generally, just as the abolition of tolls on roads and bridges has been followed by increases in traffic. The extent of the increase in prosperity resulting from such a move has been investigated mathematically by the author, using the theory of line integrals and integrability conditions. The idea that all the overhead of such an enterprise as a railroad ought to be paid out of operating revenues, when examined mathematically, is found to rest on a fallacy."

On Wednesday evening at 8:30, in the Betsy Ross Room of the Benjamin Franklin Hotel, Alexander Sachs, Director of the Division of Economic Research and Planning of the National Recovery Administration, delivered a public address on "The Postulates of Economic

Planning" which had been prepared jointly with Stephen Du Brul, the Acting Director.

Thursday morning at 10:00 o'clock, a symposium took place on the subject "Price, Supply, and Demand Phenomena."

E. J. Working of the Bureau of Agricultural Economics opened this symposium with an address[1] entitled "Studies in Demand During Periods of Rapid Economic Changes."

Dr. Elizabeth W. Gilboy of Harvard University spoke next on the subject, "Time Series and the Derivation of Demand and Supply Curves (A Study of Coffee and Tea, 1850–1930)." Dr. Gilboy said, "Statistical curves derived by graphical methods from the logarithms of price and quantity data for coffee and tea (1850–1930) illustrate the difficulty of interpreting such schedules in the light of classical equilibrium theory. Without justification of basic assumptions and verification by the economic history of the commodity, curves which are excellent statistically may be meaningless economically. Such is the case with tea, in which the statistical results find no justification in economic fact. For coffee, however, a highly inelastic demand schedule for the period 1875–1918 is corroborated by the economic history of the commodity."

The question "Is loss leader retailing ever sound economics?" was considered in a paper "Loss Leader Selling" by C. F. Roos and Henry Pixley of the Division of Research and Planning of the National Recovery Administration. They examined the practice of selling a leader or leaders at a loss in order to increase trade in other goods in a retail establishment, in the light of its effects on the merchant as reflected in his profit and loss statement, and on the consumer as reflected in the distribution and price of goods. The answer to the question depended, the speakers held, upon two factors, the total quantities of the leader and companion goods sold, and the relative sensitivities of the two types of goods to changes in price. If the first of these factors was sufficiently large, and the relative sensitivities did not differ from each other too much (Mr. Roos and Mr. Pixley give criteria for determining what the differences could be), then it was found that loss leader selling was profitable to the merchant and resulted in the consumer buying more goods. It was possible for the price of the companion goods, as well as the price of the leader, to be lowered in the process. This possibility depended on the ability of the merchant to obtain sufficient discounts for quantity purchasing when this was justified by quantity selling. If the commodity on which a loss was taken by the individual merchant happened to be one for which the total amount consumed by the community depended only slightly

[1] This paper appears in this issue of ECONOMETRICA, pages 140–151.

upon the price, the increased business was at the expense of the other merchants. However, if the commodity was one for which the community would consume appreciably more when the price was discounted, the increased sales of the merchant represented actually increased total sales in the community, although it might in addition represent shifts in sales also. In such cases, if all merchants in the community used the loss leader to the same extent, the distribution of goods, and also profits to the merchants, would increase. Thus, a power company could afford to sell certain electrical appliances below cost in order to increase the consumption of power, and need not necessarily charge more for the power in order to make greatest profit.

Messrs. Roos and Pixley concluded with the remarks that it had long been recognized that demand for goods depended on selling price, which depended on cost price, which, in turn, depended on quantity sold. They announced that a method for determining the relationships in this cycle, which had commonly been supposed to have no solution, had been developed.

Mr. Victor Perlo and Dr. C. F. Roos of the Division of Economic Research and Planning of the National Recovery Administration, next discussed the factors which determined the number of gallons of gasoline consumed by a motor vehicle in a year, in a paper "Factors Influencing the Demand for Gasoline." Charts and formulae were used to show the effects upon consumption of road building, increased or decreased purchasing power, changes in prices, registration fees, and gasoline taxes. The speakers showed that from 1925 to 1932, the continued increase in highway mileage and in technical efficiency of automobile production had resulted in a sharp upward trend in consumption per automobile. In 1930, decrease in purchasing power had checked the increase in consumption. By 1931, many car owners could no longer afford to pay registration fees, which, in general, had not been reduced. The remaining owners were those who could use their cars enough to make it worth while to pay the relatively high fixed charges. As a result, in 1931 and 1932 consumption per motor vehicle had increased while total consumption was falling off. The speakers further noted that the interrelations of the various factors differed widely from state to state. A one cent increase in the price of gasoline in Pennsylvania caused the motorist to use twelve gallons less gas in a year. In Kansas a one cent increase caused a decrease of but three gallons. An added mile of good highway increased consumption twice as much in Virginia as in Mississippi. The speakers contended that psychological factors also played a part. In 1926, the Virginia tax rate was increased by one and one-half cents. The resulting decrease in consumption had been more than twice as great as that caused by a one and one-half

cent price increase. Again, in 1928, when another one-half cent was added to the tax rate, the decrease in consumption had been 14 gallons per car, instead of the 5 gallons which one would expect from a one-half cent price increase. The same phenomenon had been noted, to a greater or lesser degree, in all the states studied. The net effect of all these factors was expressed in a mathematical formula, called the demand law for gasoline. This formula, according to the speakers, enabled one to tell to within 2 per cent the annual consumption of gasoline per motor vehicle, if one knew price, highway mileage, and the other factors involved. It could be used to determine the desirability of proposed changes in any of these factors, so far as they related to gasoline consumption.

On Thursday afternoon, at 2:30 o'clock, the Econometric Society held a joint session with the American Statistical Association, at which time papers were given under the general heading "Statistical Analysis of Time Series."

The first paper, "Analysis and Correlation of Time Series," was delivered by Professor Harold Hotelling, who said, "The extraction of valid inferences from time series, and of measurements of the strength of inference, must rest upon the same fundamental principles as statistical inference generally. The modifications of technique necessary in the case of time series are those required to obtain quantities which may properly be regarded as drawn independently at random from a hypothetical infinite population. This object is the underlying reason for the elimination of trend and seasonal variation, and of the use of smoothing and of first or second differences in the correlation of time series. The most generally applicable method of dealing with trends is by means of series of orthogonal polynomials, fitted by least squares. These have the advantages of exact interpretations in terms of probability in certain cases, the number of degrees of freedom being known; of ease of computation by summation methods, the best of which seems to be that of Aitken; of the possibility of going on to higher order terms than originally contemplated without loss of labor; and of the apportionment in a unique and simple fashion of variances and covariances into additive components, one corresponding to each component of the trend, and one to the residual deviations. The simultaneous treatment of trends and seasonals may be accomplished in the case of quarterly data by fitting a separate trend for each quarter. The seasonal index then varies continuously from year to year. Where the observations are more numerous than four to a year, and where the *continuity* of seasonal effect is to be used, it is better, instead of link relative methods, to fit each year with a few orthogonal components, harmonic or polynomial, and then to fit trends to the coefficients."

Mr. Max Sasuly of the National Recovery Administration spoke next, his paper being entitled, "Synopsis of Analysis of Interaction of Economic Variables." Where linear regression equations represented actual relations between time series variables, Mr. Sasuly said, they might be extended so as to comprise also the derivatives of these variables. This implied a system of differential equations expressing the mutual interactions. The relations thus formulated were completely analogous to the equations describing the dynamical interactions of physical systems such as electrical circuits. Corresponding to the parameters of such a system characterizing its inertia, friction, and elasticity elements, were the measured regression coefficients of the interacting time series variables. The speaker said that the system of equations thus derived was solved by well-established processes, yielding a forecasting relation for each variable. The solutions showed explicitly the effects of the interactions and of external forces, and also the discrepancy residuals resulting from inadequacy of the postulated relations.

The third speaker of the session was Mr. Victor von Szeliski, of the National Recovery Administration, who made "A Progress Report on Seasonal Variation." Mr. von Szeliski reported, in part, that "From a strictly mathematical standpoint the problem is, indeed, indeterminate, for when nothing is known save the period, the representation of the observed values by a sine-cosine series can be effected in an infinite variety of ways. Let the function be

$$y = f + s + \text{random errors}$$

where y is the observed value, f the trend-cycle component, and s the seasonal variation. We may suppose $\sum s = 0$ over the period of a year. The problem can be solved provided f have certain properties, e.g., that it is "smooth." We may take this to mean that the differences above a certain order are very small. Let capital letters denote summation over the period of a year. Then

$$Y = F + \text{random errors.}$$

In other words, the moving average of the data is the moving average of the trend-cycle. The seasonal variation problem can be solved if f can be deduced from Y. Method I is the "Graduation Method." Assume that during a given period (say 19 months) f can be closely represented by a quintic curve. Let the central ordinate of this curve be the graduated value of f for that month. We have an equation of this kind:

$$\sum_{-5.5}^{+5.5} (a_0 + a_1 x - a_2 x^2 + \cdots + a_5 x^5) = F(0) = Y(0) + \text{random errors}$$

for each of the (moving) totals, of which there are eight. The a's can be determined by least squares, and a set of weights found, to apply to the moving totals.

x	*17 mos.*	*19 mos.*	*25 mos.*
−13/2			.05054
−11/2			−.04063
− 9/2			−.05777
− 7/2		.3760	−.03299
− 5/2	1.092	−.7878	.00806
− 3/2	−3.530	−.1463	.04611
− 1/2	2.480	.5997	.06835

"Method II is the 'Expansion Method.' $f(x)$ may be expanded as a function of $f(0)$ and its differences. The series may be summed over a period of a year, and equated to $F(x)$, where F is the graduated value of Y. This leads to a set of equations which may be solved for $f(0)$ in terms of F and its even differences.

$$f^1(0) = \frac{A_0}{12} - \frac{73A_2}{12^2} + \frac{3775A_4}{12^3} - \frac{176,533A_6}{12^4} + \frac{8,006,095A_8}{12^5}$$

where

$$A_0 = 1/2(F_{-1/2} + F_{1/2})$$
$$A_2 = 1/2(\Delta^2 F_{-3/2} + \Delta^2 F_{-1/2})$$
$$A_4 = 1/2(\Delta^4 F_{-5/2} + \Delta^4 F_{-3/2})$$
$$A_6 = 1/2(\Delta^6 F_{-7/2} + \Delta^6 F_{-5/2})$$
$$A_8 = 1/2(\Delta^8 F_{-9/2} + \Delta^8 F_{-7/2}).$$

When f^1 has been determined, it is subtracted from the observed quantities y. The remainder is the seasonal variation, which may be examined for changing seasonal.

$$y - f^1 = s^1."$$

The paper by Professor Harold T. Davis of Indiana University and the Cowles Commission for Research in Economics, was the fifth of a series of investigations relating to the structure of economic series. It was entitled: "Some Statistical Aspects of Random Series." The point of departure was from observations of E. Slutsky, S. Kuznets, and others, that periods in the sense of harmonic analysis will appear in random series which have been smoothed by moving averages. This point of view had been extended by R. Frisch, who had studied the effect of various types of linear operators applied to random series. The present paper attempted to describe the Schuster periodogram of a series, $\phi(s)$, in terms of its auto-correlation function: $r(t) = \int_{-\infty}^{\infty} \phi(s)$

$\phi(s+t)ds$, by a new device. Assuming that $\phi(s)$ was an even function and expansible in a Gram-Charlier series, $\phi(s)$ was obtained in terms of $r(t)$ by an application of a theorem due to G. Pólya (*Math. Annalen*, LXXV, (1914), 376–379), and its periodogram was computed from the equation: $R(T)=\frac{\beta^n}{\sqrt{\pi}}\int_{-\infty}^{\infty}\cos\beta t r(t)dt, \beta=2\pi/T$. Applied to random series smoothed by a moving average of λ items, periods were found at .67λ and 2λ. On other hypotheses, periods had been found by Frisch at .67λ and 3.2λ, and by the author at .67λ and 2.7λ. Since none of the hypotheses was rigorously fulfilled by random series, statistical evidence must be appealed to. The harmonic analysis of four random series, smoothed by a moving average of 12 items, yielded the following periods, asterisks indicating relative intensities:

Series 1:	8*	15–17	24–26*	31*	46
Series 2:	8*	14*	21**	31*	42*
Series 3:	9*	16–18	22**	32*	
Series 4:	8**	16	22*	30*	37*

The evidence seemed to point strongly to periods at .67λ, 2λ, and 2.7λ. In order to test various hypotheses relating to the effect of erratic shocks, changing harmonics, and chance fluctuations, upon the periodicity of series, the laboratory of the Cowles Commission had performed seven extensive experiments. In one of these, 1200 items were obtained from the erratic oscillation of a galvanometer, the shocks creating the oscillations being random both as to time interval, intensity, and direction. Professor Davis presented the results of these experiments by means of lantern slides.

In the discussion which followed, Mr. Arne Fisher criticized rather severely several of the conclusions. On the one hand he took issue with Professor Hotelling's remarks on "statistical inference" especially in the estimating of certain semi-invariants by means of R. A. Fisher's "cumulants" or "cumulatives," which latter terms Mr. Fisher referred to as "philological masochisms." In the way of a test of the reliability of certain estimates, the speaker suggested that a substitution of the "cumulants" K_4 and K_2 for the ultimate (or true) semi-invariants λ_4 and λ_2 in the *a priori* formula

$$\lambda_2(\mu_2) = n^{-3}(n-1)\lambda_4 + 2n^{-2}(n-1)\lambda_2^2$$

of a sample of size n *should always yield a positive result* as otherwise the variance (or $\sqrt{\lambda_2(\mu_2)}$) would be an imaginary quantity. In several numerical illustrations of small samples of four or five observations, Arne Fisher showed that R. A. Fisher's fourth order "cumulant," K_4,

did not satisfy this test. The speaker also criticized the concept of the so-called "normal" or straight line trends in time series, and read the following excerpt from some of his own writings: "The idea that business shall conform to that quaint 'tutti frutti' known as the 'normal' and be made to move as the crow flies, in the rectilinear and narrow path of a straight line or a low degree parabola, is a charming, improbability like that other imbecile, although equally charming proposition that all men are created equal. Politicians, executives, and economists, who gabble on business cycles as something abnormal that should be avoided, might do well to bear in mind that space is full of pulsations. Any biological (economic) force, nay even any force in nature, such as heat, light, or sound, that travels far and with a moderate or high velocity does not move in the direction of itself but nearly at right angles to itself, so that its path of motion takes the form of oscillating waves. From this point of view, the idea of the purely postulated straight 'normal' trend goes all to pot, and the business cycle might well be regarded as the truly normal movement."

On Friday morning at 9 o'clock, a session was held for the discussion of "Income and Purchasing Power."

Mr. Victor von Szeliski read the opening paper, "Frequency Distribution of National Income." Mr. von Szeliski showed that for incomes above $5000 the frequency distribution $y(x)$ as determined from Statistics of Income data, followed closely Pareto's law, $y=\chi^{-\alpha}\cdot|\alpha|$ varied inversely to the per capita real income.

1916	2.33	1928	2.43	1930	2.60
1921	2.70	1929	2.45	1932	2.70 E

Mr. von Szeliski indicated that there were no data from which the frequency distribution below $5000 could be directly determined. However, it could be built up out of frequency curves for wage earners and others. For example, the speaker said "suppose that y for wage earners is given by $w=Ax(u-x)^a$. u may be taken at $3500. The number of wage earners $N=A\int_0^{3500}x(3500-x)^a dx$ and their total income $I=A\int_0^{3500}(u-x)^a dx$. N and I are known (e.g. from King's *The National Income*) and consequently A and a can be determined. In the table below for $10,000 and over, the figures are from Statistics of Income, adjusted to eliminate speculative gains and losses. Below $10,000, the frequencies represent wage earners calculated as above, plus non-wage earners determined by extrapolating $y(x)$ for $\chi\geqq\$10{,}000$ back to $\chi=0$, so as to reproduce (1) the number of individuals in this group, and (2) their income.

"The validity of the above method remains to be proved by checking against an actual Census of Incomes. However, the error involved is

probably of the order of that in the national income figures themselves. On the basis of the above table, it appears that, while the soap-box slogan that '5 per cent of the people control 90 per cent of the wealth' may not be statistically correct, 5 per cent of the income recipients get between 26 per cent and 27 per cent of the national income. One per cent get about 16 per cent, 1/10 of 1 per cent about 8 per cent, and 1/100 of 1 per cent about 4 per cent, of the national income."

INCOME DISTRIBUTION BY INCOME CLASSES
1929

Income Class	Number in Each Class	Cumulative	Per cent	Income in Millions	Cumulative	Per cent
Under $ 1,000	15,472,560	48,500,000	100.0	$ 9,567	$90,500	100.0
$ 1,000– 2,000	20,117,510	33,027,440	68.1	29,487	80,933	89.4
2,000– 3,000	8,962,940	12,909,930	26.6	21,462	51,446	56.8
3,000– 4,000	1,994,920	3,946,990	8.13	6,773	29,984	33.1
4,000– 5,000	720,210	1,952,070	4.02	3,216	23,211	25.6
5,000– 10,000	770,909	1,231,860	2.54	5,339	19,995	22.1
10,000– 25,000	339,871	460,951	0.950	5,032	14,656	16.2
25,000– 50,000	77,039	121,080	0.250	2,623	9,624	10.6
50,000– 100,000	28,021	44,041	0.0909	1,908	7,001	7.73
100,000– 250,000	11,648	16,020	0.0330	1,749	5,093	5.63
250,000– 500,000	2,842	4,372	0.00891	911	3,334	3.70
500,000– 1,000,000	973	1,530	0.00316	663	2,433	2.68
1,000,000 and over	557	557	0.00115	1,770	1,770	1.96

The second paper on "Purchasing Power" was prepared by Stephen Du Brul and Andrew Court, of the National Recovery Administration.

Professor George Bousquet of the University of Algiers, Algiers, North Africa delivered the third paper of this session on the subject, "The Velocity of Circulation of Capital." In summary, Professor Bousquet said, "This problem never has been studied except by K. Marx, and his ideas are almost wholly wrong. This magnitude is of dimension T^{-1}, it is namely: $\frac{\text{a Flow of value}}{\text{a Stock of value}}$ in the unit of time. As to the numerator, one must try to take account of the interest paid to the capitalist during the time that the capital goods remain in the enterprise. As to the denominator, it is shown that logically we must consider also the stock value of the perpetual (natural) capital goods, and even of the wage earners. The proposed formula is $V=\frac{\sum(k/t)C}{\sum C}$, t is the time during which we need to use a capital good (stock) in order to produce one unit of product, and k the fraction of the capital good used in the time t. If p_0 be the price at which an enterprise buys its capital goods, we put $C=p_0(1+\frac{it}{2k})$; i = rate of interest."

The subject "Money" was canvassed at a joint meeting of the Econometric Society and the American Statistical Association on Friday

Morning, December 29, at 10:30 o'clock. Professor Irving Fisher of Yale University presided.

Dr. Robert Eisler of Vienna contributed the first paper, taking as his subject "The Commodity Dollar."

Dickson H. Leavens, formerly Treasurer of Yale-in-China, then spoke on "Silver in the Monetary System." "Agitation to do something for silver comes largely from interested sources," said Mr. Leavens. "The argument for increasing foreign purchasing power ignores the fact that only China is on a silver standard, and that in her case the fall of silver has facilitated price adjustments. The argument for increasing the monetary base is weakened by the limited amount of silver that would be available, even if raised to the old 16 to 1 level. The argument for silver as a means of inflation neglects other more practicable means of inflation. President Roosevelt's silver coinage plan of December 21 will have little effect except on American silver producers, but may be a good political move."

The session of Friday afternoon was devoted to the problem of "Index Numbers."

Dr. Isador Lubin, of the United States Bureau of Labor Statistics, dealt with the topic "Statistical Requirements in the Field of Labor."

Mr. Herbert Harvey of the Lehman Corporation spoke on "Index Numbers and Measurement." Mr. Harvey stated that the distinction between the indefinite and the definite quantity separated the proper fields of index numbers and measurement. But contemporary terminology ignored the distinction and betrayed the confusion of thinking that refinement of indexes constituted measurement. Measurement set out, Mr. Harvey said, to obtain from observations the most probable or median probable value of a defined quantity and its probable error. The classical theory of measurement was virtually limited to the Gaussian distribution of errors and was ineffective in economics. The speaker proposed a family of distributions with a wide range, resulting in a form of average intermediate between the mean and the median.

* * *

The program of the Econometric Society Meeting in Boston consisted of a single session, which was held jointly with Section K of the American Association for the Advancement of Science, on Saturday, December 30, at 10:00 A.M. in Lawrence Hall of Harvard University. The attendance was small but select.

Charles F. Roos and Victor von Szeliski first presented a paper "Analysis of Causes of Fluctuations in Building Activity with Special Reference to the Present Economic Situation." They reported that they had been able to subject the principal factors affecting new resi-

dential building to mathematical analysis and to find statistical measures of the relative importance of the factors—rental income, taxes, replacement costs, occupancy, population growth, and financing. They exhibited charts and a formula used in resolving these factors, which gave estimated residential building almost equal to actual building in St. Louis year by year for the last thirty-three years. The speakers said that "this formula states simply that part of the new building is due to expected income, and part to other causes, for example, building by people of means for self-gratification, etc. These other causes, however, appear to be remarkably constant and variations due to them can be safely neglected. Many who have incentives to build for income purposes are discouraged by financing difficulties. The influence of a change in the foreclosure rate, which in the case of St. Louis measures the ease of financing, is apparently negligible for rates of foreclosure greater than 500 per year per 100,000 families. Investors are hardly more inclined to advance money when the foreclosure rate equals 500 than when it equals 2000, as now. When the foreclosure rate drops below 500, the inhibitions on capital flow disappear, and if it drops to 200, capital flows freely indeed. It must be remembered, however, that foreclosures as used here measure the availability of credit; prohibiting foreclosures does not release credit.

"A doubling of the income incentive, rent multiplied by percentage of occupancy minus taxes divided by replacement cost, will not quite cause a doubling of the volume of building. More building results from raising a low income incentive to a moderate incentive than from raising a moderate incentive to a high one. Halving the cost of building will produce only 86 per cent more building due to income incentive.

"Building is inelastic in relation to cost because quite a volume of building takes place without regard to cost. The proper way to measure cost is in relation to net rentals. Although rents did start upward in 1918, 1919, and 1920, costs rose so much more rapidly (especially in 1920) that demand for dwellings was choked off. The high costs of 1924 and 1925 (as high as in 1920) did not adversely affect building, because in the meantime rents had risen. Most building takes place when cost is equal to ten times net rentals. But credit conditions are more important than net rentals alone.

"Residential building in St. Louis proper has gone through two major cycles since 1900. Although complete data are not available for other cities, it is believed that similar cycles will be found. In St. Louis, as for the United States as a whole, physical building needs, as calculated from the number of families, have been stable, but with a declining tendency. This tendency has been accentuated with the slowing up in population growth. The downward movement of rents, which

started in 1926, has been aggravated by increasing vacancies and mounting taxes. Net rent after deduction of taxes is back to pre-war levels. Variations in net rents and credit conditions tend, on the average, to be followed by changes in volume of building within two or three years. This makes it possible to estimate new residential building for next year and the year after. Unless there are important changes in occupancy, rents, replacement costs, taxes, and especially credit facilities for financing homes, there will be a continuation of present low levels in residential building in the next two years. Occupancy can be increased by unscrambling doubled-up families (increased employment), by demolishing obsolete, unsanitary buildings that are now being rented, and by an increase in the marriage rate, which is now far below normal.

"Detailed analysis shows that if credit is made easier, and building costs (freight rates, wage rates, material costs, etc.) are kept at reasonable levels, a recovery in residential building to about one-third of the peak can be expected next year. Under such conditions, residential building in 1935 could reach two-thirds of the peak volume.

"In view of the fact that three to five million workers are normally engaged in residential building, and several million others are indirectly dependent on residential building, and these are practically all without jobs now, too much importance cannot be attached to the problem of reviving residential construction."

Roos and von Szeliski concluded that the significant fact about their study was that it apparently succeeded in revealing causal sequences which preceded the event by about two years. This had been accomplished by a difficult but unavoidable mathematical analysis. If this newly developed technique succeeded for other industries, a signal contribution had been made to business analysis.

Dr. Eugen Altschul, formerly Chief of the Business Cycle Institute, Frankfurt, Germany, gave the next paper on "Probability and Statistics." Dr. Altschul held that the empirico-statistical studies of the last ten years had not enriched our economic knowledge as much as many investigators had expected. Disappointment, similar to that which followed the first apparent successes of the Historical School, now threatened us. Statistics could never be substituted for economic theory. On the contrary, only theoretical analysis provides the assumptions and checks necessary to a correct application of mathematical statistical methods. Any fruitful use of these methods, Dr. Altschul felt, must be based on a probability scheme order model like that derived from games of chance. The establishment of a relation between the empirical frequency distribution and such a model was the primary logical requirement. Therefore, the rules of applied mathematics de-

manded an analysis of empirical series based upon definite assumptions, that is, on a hypothetical scheme characteristic of the causal universe underlying the empirical phenomenon. Which hypothetical scheme or model was adequate could not be decided directly upon an empirical basis. Theory must uncover what specific assumptions were to be made. Only when this condition was satisfied, Dr. Altschul concluded, would statistical methods be more than a tool to describe historical facts: they would then serve in the discovery of causal relations.

A paper "The Adequacy of 'Student's' Criterion of Deviations in Small Sample Means" by Alan E. Treloar and Marian Wilder of the Biometric Laboratory of the University of Minnesota, concluded the session. In summary, the speakers said, "The conventional test for the significance of the deviation of a sample mean from a given population or 'supply' value (based upon the probability of the ratio, z, of that deviation to the standard deviation of the sample itself) has been investigated independently by Karl Pearson and by Egon Pearson in recent years in a series of critical studies. The work of the former is extended in this paper to an enquiry into the magnitude of error involved in assuming the 5 per cent and 1 per cent levels of significance of the z distribution to indicate significant deviations in the means alone. The necessary probability integrals have been evaluated for a series of values of N odd, and the results verified for $N=5$ by two extensive sampling tests. The criticisms of Karl Pearson relative to small samples are sustained and extended to samples previously considered to be large enough to yield dependable results."

ALFRED COWLES 3RD

Cowles Commission
for Research in Economics

ROBERT SCHMIDT'S DEFINITION OF SKEWNESS AND KURTOSIS

SKEWNESS and kurtosis may be defined in different ways. In a paper to be published elsewhere, Dr. Robert Schmidt gives the definition by means of the n-iles of the distribution. The following is a somewhat modified and condensed version of his developments, published here with the author's kind permision.

Let x be a quantity of which a great number of observations are made. The frequency distribution of these observations may be defined by indicating n points, $x_1, x_2 \cdots x_n$, on the x axis such that $1/2n$ of the observations fall to the left of x_1, $1/n$ of the observations fall between x_1 and x_2, $1/n$ between x_2 and x_3, and so on, and, finally, $1/2n$ to the right of x_n. The numbers x_s defined in this way may be called the n-iles of the distribution. As n increases, the set of numbers x_s defines the distribution with an increasing accuracy. The idea of characterising the distribution by its n-iles is obviously equivalent to considering the inverse of the function $F(x)$ which indicates how the total number of observations equal to, or less than, x varies as a function of x.

This being so, if one wants to compare two distributions, a natural idea seems to be to do so by comparing their n-iles. For instance, if y_s are the n-iles in a given distribution, and x_s the n-iles in some standard distribution with which the first distribution is compared, and if $y_s = x_s$, then the distribution studied is equal to the standard reference distribution, at least in so much detail as is expressed by the n-iles.

If y_s is not equal to x_s, it may be a simple function of x_s, possibly a polynomial in x_s (with coefficients independent of s), or at least y_s may be approximated by a polynomial in x_s. Such an approximation is the leading idea of Schmidt's definition of skewness and kurtosis. These notions for the distribution y_s (as compared with the given distribution of comparison x_s) are defined by the coefficients of the polynomial by which y_s is expressed as a function of x_s.

As is well known, the various formulae of polynomial approximation turn out in a more convenient form if the approximations are ordered in terms of a set of orthogonal polynomials instead of in terms of the powers themselves. This applies in particular to the present problem. Therefore let us consider such a system of polynomials:

$$(1) \quad z_{hs} = z_h(x_s) = b_{h0} + b_{h1}x_s + \cdots b_{hh}x_s^h \qquad (h = 0, 1 \cdots n-1)$$

The fact that these polynomials ought to be orthogonal (and normalised) is expressed by the condition

$$\frac{1}{n}\Sigma_s z_{hs}z_{ks} = e_{hk} \tag{2}$$

where e_{hk} are the unit numbers, that is to say, equal to unity if $h=k$ and equal to 0 if $h\neq k$, and where the summation is extended to the whole set of n-iles, that is to say $s=1, 2 \cdots n$.

The condition (2) determines the coefficients b uniquely. Indeed putting first $h=0$ we get from (1) $z_{0s}=b_{oo}$, inserting in (2) and putting by convention the square root positive, we obtain

$$b_{00} = 1. \tag{3}$$

Next putting $h=1$ and inserting in (2) using (3) we get the two equations in b_{10} and b_{11}.

$$n\,b_{10}^2 + 2b_{10}b_{11}\Sigma x_s + b_{11}^2\Sigma x_s^2 = n$$
$$n\,b_{10} + b_{11}\Sigma x_s = 0.$$

The summations in these equations are to be understood as summations extended over all the n-iles of the distribution x_s. The above equations give

$$b_{11} = 1/\sigma_x$$
$$b_{10} = -\,\bar{x}/\sigma_x \tag{4}$$

where

$$\sigma_x = \sqrt{(x_s - \bar{x})^2/n} \tag{5}$$

$$\bar{x} = \Sigma x_s/n \tag{6}$$

$\bar{x}$ is the mean of the n-iles x_s and σ_x their standard deviation. Continuing in this way all the coefficients of the transformation (1) may be determined up to any order.

The numbers z_{hs} defined by this process may be called the orthogonal n-iles. By virtue of (3) and (4) their expressions in terms of the original n-iles x_s are

$$z_{0s} = 1$$
$$z_{1s} = (x_s - \bar{x})/\sigma_x \tag{7}$$
$$\cdots \text{etc.} \cdots$$

In other words, z_{0s} is a constant and z_{1s} are the n-iles measured from their mean and reduced by their standard deviation. In a similar way the higher orthogonal n-iles may be determined.

From (7) it is obvious that the first orthogonal n-iles z_{0s} and z_{1s} do not change if the x_s are subject to a linear transformation (the same for all the x_s). And this proposition holds for the z_{hs} of any order.

By (2) we see that the square of the determinant of the first n orthogonal n-iles is a diagonal determinant with non-vanishing elements, hence non-vanishing. Consequently the system of linear equations

$$(9) \qquad y_s = a_0 z_{0s} + a_1 z_{1s} + \cdots + a_{n-1} z_{n-1,s} \qquad (s = 1, 2 \cdots n)$$

where $y_i \cdots y_n$ is a set of arbitrary numbers, has a unique solution in $a_0 \cdots a_{n-1}$. In other words, if $y_1 \cdots y_n$ is any set of given numbers, this set may be expressed as a linear form in the orthogonal n-iles z_{hs}. This holds good in particular if the numbers $y_1 \cdots y_n$ are interpreted as the n-iles of a given distribution of comparison x_s.

The coefficients a_h of (9) may easily be expressed directly in terms of the two sets of n-iles y_s and x_s. Indeed, since the set z_{hs} is orthogonal, we get immediately by multiplying (9) by z_{hs} and performing a summation over s

$$(10) \qquad a_h = \Sigma z_{hs} x_s / n \qquad (h = 0, 1 \cdots n-1)$$

The coefficients a_h may be called the Tchebycheff coefficients of the distribution y_s (with regard to the reference distribution x_s).

Inserting (7) in (10) we get

$$(11) \qquad a_0 = \Sigma y_s / n$$

$$(12) \qquad a_1 = \frac{1}{n} \frac{\Sigma x_s y_s - n \cdot \bar{x}\bar{y}}{\sigma_x} = \frac{\Sigma (x_s - \bar{x}) \cdot (y_s - \bar{y})}{n \cdot \sigma_x}$$

where $\bar{y} = \sum y_s / n$.

The numbers $(x_s - \bar{x})$ obviously form a non-decreasing sequence. And if the set y_s is the set of n-iles for a given distribution, the set $(y_s - \bar{y})$ will also form a non-decreasing sequence. Consequently, the two sets $(x_s - \bar{x})$ and $(y_s - \bar{y})$ are monotonic and change in the same direction and have both the mean 0, but then it is an immediate consequence from the classical Tchebycheff inequality that the right member of (12) must be non-negative.

The coefficient a_0 is the mean of the n-iles in the distribution y_s and the coefficient a_1 may be called the standard deviation of the set y_s (taken with regard to the reference set x_s). In a similar way, the higher coefficients a_2, a_3, etc., may be determined. The skewness S and kurtosis K of the distribution y_s (with regard to the distribution x_s) may then be defined as

$$(13) \qquad S = \frac{a_2}{a_1}$$

$$K = \frac{a_3}{a_1} .$$

EDITOR

THE MEETING OF THE ECONOMETRIC SOCIETY, BERKELEY, CALIFORNIA, JUNE 1934

PRELIMINARY arrangements have been made for a meeting of the Econometric Society which will be held in Berkeley, California, on June 21, 22, and 23, in connection with the meeting of the American Association for the Advancement of Science. The headquarters of the Econometric Society will be maintained in the Stevens Union Building on the campus of the University of California.

A local branch of the American Statistical Association and the Social Science Research Conference of the Pacific Coast have been invited to hold one or more joint sessions with the Econometric Society.

The tentative arrangements which have been made indicate that papers will be given by:

Dr. Fritz Machlup of Vienna, Austria
Dr. George Peterson of the University of California
Dr. Leo Rogin of the University of California
Professor E. L. Grant of Stanford University
Professor John B. Canning of Stanford University
Dr. Holbrook Working of Stanford University
Mr. O. P. Wheeler of the Federal Reserve Bank of San Francisco
Dr. Norman J. Silberling of San Francisco
Professor Raymond Garver, University of California at Los Angeles
Dr. Lewis A. Maverick, University of California at Los Angeles
Mr. W. F. C. Nelson, Cowles Commission for Research in Economics, Colorado Springs.

Further additions of other speakers are now being arranged.

It is requested that all members of the Econometric Society who expect to attend the meeting at Berkeley should write to Professor John B. Canning or Dr. Holbrook Working, both of Stanford University, Palo Alto, California, who are in charge of the program.

JOHN B. CANNING

Stanford University

William Stanley Jevons
1835–1882

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WILLIAM STANLEY JEVONS

I

HIS LIFE

By H. Winefrid Jevons

Practical experience in the making of a currency is not a common part of the training of an economist, yet it may perhaps be a valuable one. William Stanley Jevons at the age of 18 was appointed one of the two assayers at the newly opened Sydney Mint. The work was offered him through the chemistry lecturer at University College, London, where he was a student, and he took the post with reluctance, owing partly to his desire to continue studying, and partly to the great distance from home; the voyage out took three months in a sailing ship. His father, Thomas Jevons, was an intellectual and able man who was interested in many engineering schemes, and in 1815 probably invented the first iron boat to sail on water. He was occupied in the family business of iron merchants in Liverpool, a development from an early nail-making business in Staffordshire, and owing to financial reverses it became necessary for Stanley to obtain some paid work. He accepted the Sydney post at his father's wish and not his own, yet the years spent in Australia were undoubtedly helpful in giving him time to shape his own plans and to realize where his great interest and talent lay. At least it was better than his own first idea of work in a Liverpool broker's office with two or three hours of study each night.

So far, Jevons' interests had lain in the natural sciences. Watching the newly made sovereigns pass into circulation must have drawn his attention to currency questions, while visits to the gold fields and studies from the industrial and social standpoint of the rapidly growing Australian towns were excellent experiences for one whose thoughts in later life were absorbed by economic problems. The practical work of assaying, too, was a training in accuracy. It may well be that when Jevons wrote his first well-known pamphlet in 1863, *A Serious Fall in the Value of Gold ascertained and its Social Effects set forth*, his thoughts turned to the assayers' room at the Sydney Mint and to the diggers at Ovens and the other big gold fields whom he had so well described in letters home, together with the crowds of dealers, storekeepers, and traders of all descriptions, who had flocked there to exchange produce for the diggers' gold.

That metal made no appeal to Jevons for its own sake. The position at the Mint was a well-paid one, and after four years there he determined to return to England, obtain some further study for a year by

utilizing his savings, and then chance what might follow after, with a possibility of returning to Australia if England offered no suitable means of livelihood. First, however, he saw more of the world by travelling home through South and North America and the West Indies, all voyages being again by sailing ship. Travel was always a delight to him.

During Jevons' time in Australia his father had died; his mother, a daughter of the well-known Liverpool citizen, William Roscoe, had died when he was only ten years old. He returned, therefore, to share lodgings in London with his sisters and brother, for whom he was to some extent responsible, and from there entered University College once more, at the age of twenty-four, taking his B.A. degree in eleven subjects at the end of a year, and so attaining what had been the height of his ambition when in Sydney. He adhered to his decision, made in Australia, to apply himself to a study of "the foundations and nature of the knowledge of man" to which he felt he had a mission, giving considerable attention to political economy during his B.A. work. In June 1860 he wrote to his brother in New Zealand: "I have fortunately struck out what I have no doubt is the true Theory of Economy . . . while the theory is entirely mathematical in principle, I show at the same time how the data of calculation are so complicated as to be for the present hopeless. . . . Most of the conclusions are, of course, the old ones stated in a consistent form, but my definition of capital and law of the interest of capital are, as far as I have seen, quite new." It is amusing to find that he had expected to be first in the class examination owing to having given so much thought to the subject and read widely on it, but met with "a sad reverse" being placed third or fourth, a fact which he attributed—probably rightly—to a difference of opinion between himself and the examiner. He consoled himself with the thought of a pleasant revenge when he published his "Theory of Economy" and re-established the science on a sensible basis.

The award of the Ricardo Scholarship was a consolation, and this, with some desultory writing, chiefly on scientific subjects, enabled him to spend two more years at the College, taking his M.A. degree in Mental Philosophy and Mathematics in June 1862, and winning the gold medal awarded to the best student in that class. Mental philosophy in those days included the subjects of Logic, Political Economy, and Philosophy, including works of the Greek Philosophers. In Mathematics, as in his B.A. work, he had the great advantage of attending classes taken by De Morgan, "an unfathomable fund of mathematics" as he described him. During his first year's work for his M.A., Jevons conceived the idea of publishing "a statistical atlas" which should give in about thirty plates of curves and diagrams all the available statistics of trade, finance, wages, pauperism, and other matters, showing the

state of the country and its people from as far back as figures were available. Much time was spent at the British Museum compiling data, only to find that no publisher would sponsor the idea, and the project ended in his publishing at his own expense in 1862 just two diagrams, the first showing the weekly accounts of the Bank of England, and the other the price of the English funds and of wheat, the number of bankruptcies, and the rate of discount.[1] These received some favorable mention.

In September 1862, he sent two papers to be read at the meeting of the British Association, the first entitled *Notice of the General Theory of Political Economy* and the other *On the Study of Periodic Commercial Fluctuations, with five diagrams.* He made a note that the Secretary of Section F informed him that the second was approved of.

Following his M.A. degree, Jevons tried a year in London, earning precariously by writing and by acting as searcher in the British Museum. His services there were not much requisitioned, and writing did not prove a paying profession. This is not to be wondered at, perhaps, as he was giving his time to the pamphlet on gold already mentioned, which was published that spring; and in December 1863 he published a work on *Pure Logic or the Logic of Quality apart from Quantity, with Remarks on Boole's System and on the Relation of Logic and Mathematics.* By that time, however, Jevons had had to relinquish his idea of living by writing, and had accepted a post as tutor at Owens College (now the University of Manchester) at the instance of his cousin, Henry Roscoe, the chemist, who was already installed there as one of the six professors. The post was a poorly paid and arduous one, for the tutor was expected to coach students in any subjects they required, and to hold classes in mathematics, Greek, and Latin. An intense shyness and dislike of public speaking made the work still more of a strain for one who was happier by far at that stage in working at his own theories in solitude. He held this tutorship for two years, however, but could not refrain from pursuing his own work at the same time. The long vacation of 1864 was entirely devoted to the problem of the exhaustion of England's coal reserves, *The Coal Question* being published in April 1865 and first making him "a known man." Already he had been elected a fellow of the Statistical Society and had spoken at the meetings of the Manchester Statistical Society. In 1864 he noted in his journal how his wildest dreams in Sydney had been exceeded, what respect he then felt for the name "statistician," and how reviews of authority had now called him "a competent statistician."

[1] These diagrams were reproduced in the first edition of *Investigations in Currency and Finance.*

The Coal Question, on account of its practical importance and the breadth of vision displayed by its author, made a more direct appeal than the works on the value of gold and on logic. Mr. Gladstone, then Chancellor of the Exchequer, was much interested, and used the probable diminution of our most advantageous coal supplies as an argument for a large reduction in the national debt, and Jevons found himself quoted more than once in parliamentary debates. As a result of this book the Royal Commission on the Coal Supplies was appointed.

It was, I think, before this book was known that the college of his native town, Liverpool, gave him the honor of his first professorship in Logic and Political Economy. The work was not much, it involved one night a week away from Manchester, but it was in his particular line. The same year he was able to give up the tutorship at Owens College, and was appointed lecturer in political economy and substitute lecturer in logic. This was followed next year by election to the newly created professorship of logic and mental and moral philosophy, and the Cobden Professorship of Political Economy at Owens College. That, even then, my father had no realization of the position he had won, is shown by the extreme anxiety with which he awaited the decision about this appointment. The relief of obtaining a comparatively well-paid post, allowing some leisure for his own work, was great.

During all these years, my father's thoughts seem to have turned naturally from economic questions to logic and then back again. After *The Coal Question* was out, the invention of a logical machine absorbed much of his time, and finally a second and greatly improved machine was shown to a meeting of the Royal Society in 1870 and a paper read thereon. This machine is preserved in the Science Museum at South Kensington, London.

In 1869, Jevons published *The Substitution of Similars, the True Principle of Reasoning, Derived from a Modification of Aristotle's Dictum*, and the following year his most widely known book on *Logic, the Elementary Lessons*.

Taxation and Currency problems were not neglected, since the list of his writings shows papers and letters written on these questions of the moment, and in 1870 he acted as president of Section F (Economic Science and Statistics) of the British Association meeting at Liverpool, his address being on rather varied social questions such as the connection of poverty and indiscriminate charity, the need for self-reliance amongst the poor, the incidence of taxation, and the excessive mortality in some of our big towns.

Jevons, as already mentioned, first conceived his new views on the theory of Political Economy while still a student; and when he sent a paper on it to the British Association Meeting in 1862, a very brief

abstract was inserted in the Annual Report. The original paper appeared in the *Journal of the Statistical Society* for 1866; but it was not till the books on logic were completed that he set to in earnest to elaborate in book form his theory of utility and the mathematical approach to economics. He spent some eight or ten months over writing *The Theory of Political Economy*, which was published in 1871.

M. Léon Walras of Lausanne, unaware of the earlier publication of this theory, had arrived at much the same views independently, and in 1874 some correspondence between them occurred as to who was first in the field; M. Walras, on seeing the *Statistical Journal* for 1866 generously acknowledged my father's priority and published the correspondence in the *Journal des Économistes* under the title "Théorie Mathématique de l'Échange, Question de Priorité," though as my father wrote, "As to the question of priority of publications, it is of course, of less importance than that of the truth of the theory itself." In England, the mathematical theory of economics had, up to that time, been either neglected or criticized adversely, many economists not having sufficient mathematical grounding to follow all the arguments in the book; and Jevons regarded Walras's independent writings as satisfactory proof of the validity of his own application of mathematical principles, though they followed a different course of reasoning. Interest in the mathematical approach to political economy was indeed more advanced then on the Continent than in England, and my father corresponded about this time with various economists abroad on points connected with the *Theory of Political Economy*, and had the pleasure of seeing it translated into Italian in 1876.

In 1871, however, his thoughts had turned to logic again, and the two volume book *The Principles of Science*, his biggest work on logic, appeared that year. Here he criticized some of J. S. Mill's views, as he had done in political economy, and incurred again much opposition from the ardent followers of the latter.

Money and the Mechanism of Exchange appeared in 1875, and was translated into French, German, and Italian. It is descriptive of the monetary and banking methods of the world, and the laws of circulation, and was written clearly and simply for the untrained reader. In the same year Jevons raised the question of the possible dependence of harvests on the state of the sun as evidenced by sunspots, reading a paper at the British Association Meeting on "The Solar Period and the Price of Corn." This was his first mention of the question of sunspots and trade; but, as he then pointed out, he did not assert the truth of the connection and the paper was afterwards withdrawn for further consideration. Three years later the subject was followed up by a paper on "The Periodicity of Commercial Crises and its Physical Explanation,"

at the meeting of the British Association and by articles in *Nature*. Two more works on logic were completed, viz., a little primer, and *Studies in Deductive Logic;* and in 1882 a small book *The State in Relation to Labour* for the English Citizen Series.

The early work at Owens College had been laborious, and in addition Jevons had worked extremely hard at his own writing. The inquiry for *The Coal Question*, for instance, was only fully commenced in January 1864, and the writing was finished before Christmas, sometimes five or six hours at a stretch without a break being given to it in vacations and much being done late into the night. When not otherwise occupied, his mind was always dwelling on some economic or logic problem. The professorship too, involved some fatiguing evening work. Small wonder, then, that his health soon began to give trouble, sleeplessness and nervous and physical exhaustion inevitably following, so that when only thirty-six years old he was obliged to give up all work for a time; in 1873 he had a year's leave from the College, part of which was devoted to writing and part to tours abroad. Visits to Norway, of which he made many, were always a delight to him. Some quiet fishing and travelling in out-of-the-way parts by cariole, local steamer, or rowing-boat, gave complete mental change. It was, indeed, only by holidays taken on the move—sightseeing, driving, or, when fit enough, walking, that he was able to obtain mental rest. From this time his work at the College was lightened by the appointment of a lecturer for the evening work; but in 1875 he sent in his resignation. His reasons for taking this step were to secure leisure for writing, and the opportunity of residence in London, which would enable him to attend the Political Economy Club, where he wished for more opportunity to express his views as against the followers of J. S. Mill, and he wished also to be present at the meetings of other learned societies. The professorship in Political Economy at University College fell vacant after this decision was made. As it involved but little lecturing work, Jevons was induced to accept it, and moved to London in 1876, finding a house near Hampstead Heath which gave him the essential quiet. He resigned this chair in 1880 and also his post as one of the secretaries of the Statistical Society, as his health still gave him much trouble, and he wished to devote his time to a large work to be called the *Principles of Economics* which should deal in full with the whole science according to his methods of approach. This book was never finished. A fragment consisting of several chapters and parts of others was published as late as 1905 under the title *The Principles of Economics, a Fragment of a Treatise on the Industrial Mechanism of Society and other papers*, with a preface by Henry Higgs. Another posthumous volume of papers under

the title *Investigations in Currency and Finance* had been previously edited by Professor Foxwell.

Jevons was drowned while bathing on the south coast of England in August 1882, the shock of the cold water proving too much for his enfeebled health. He was a few weeks short of forty-seven years of age. He left a wife who had been a constant companion and help in his work, and three small children, too young to understand its nature. The widespread expressions of regret proved the esteem in which he was held by people in diverse circles and the world wide reputation he had established as a writer on logic and economics. Former pupils have shown by their remarks, however, that his college work was never neglected for his writing, and that he was held in high estimation, not only for the instruction he gave, but for his high-minded and lovable personality.[2]

II

HIS SCIENTIFIC CONTRIBUTIONS

By H. Stanley Jevons

Jevons' early publications were almost entirely on natural science. In Australia meteorology was his hobby, and for two or three years he made daily observations of pressure, temperature, wind, and rainfall, which were published in the local press, and wrote papers on such subjects as "The Deficiency of Rain in an Elevated Rain-Gauge." His growing interest in social problems and determination to be acquainted with the facts is well illustrated by a social survey of Sydney which he made during his last two years of residence there.[3] His interest in natural sciences never waned. Geology and botany provided outdoor interests; and not long after *The Coal Question* was published he purchased a good microscope, and used it, amongst other things, for an investigation of the molecular motion in liquids which he named "pedesis," but which is now generally known as Brownian Motion.

It was Jevons' training in natural science which determined his method in logic and in economics, and perhaps made him a follower of Bentham. He was excited by Mill's inconsistences, fallacies, and want of system, both in the latter's *Logic* and in his *Political Economy*, particularly by his one-sided and inaccurate presentation of the theory of value; and he felt that it was his special mission to destroy the al-

[2] I am indebted for many details in this article to the *Letters and Journal of William Stanley Jevons*, edited by his Wife, published by Messrs Macmillan & Co. in 1888.

[3] Published in the Sydney *Morning Herald*, 1929.

most universal acceptance in England of Mill's logic and political economy. His interests were wide, however, as may be realized when it is found that his writings, besides those on natural science already referred to, fall into six classes: (1) on logic and the theory of probability, (2) on prices and periodic commercial fluctuations, (3) on the pure theory of economics, (4) on money and banking, (5) factual studies, such as the social survey already mentioned and the *Coal Question*, (6) studies on social reform and labor questions.

Jevons' writings on logic are noteworthy for their clarity and conciseness, and their wealth of illustrations He was much influenced by Boole's mathematical logic, but he found much in it to criticize, and stripped it of its mathematical dress, and simplified Boole's methods. He also acknowledged indebtedness to De Morgan and Archbishop Thomson.[4] Jevons reduced logical inference to a simple but complete system, and defined the inductive or scientific method, showing its unity in all sciences, and the fundamental importance of the theory of probability.

His publications on prices and commercial fluctuations begin (except for the two diagrams mentioned in the previous section) with the *Serious Fall in the Value of Gold*, which is notable for his compilation of index numbers of prices using the geometric mean, and for setting forth how different sections of the population are affected by a rise or a fall of the price level. Now that index numbers and changes of price level are commonplaces which everybody understands, it is difficult to realize the originality of the research embodied in this pamphlet. This was followed a few years later by a statistical study of the returns of the Bank of England, which showed clearly an annual and quarterly periodicity, the former due mainly to finance of the harvest and to holiday expenditure, the latter to the customary quarterly disbursement of dividends. For demonstrating that the price level had risen after the discoveries of gold in California and Australia, it was only necessary to investigate prices from 1845 to 1862; but he subsequently extended his series of index numbers of prices back to 1782, and made a further study of commercial crises. He was struck by the frequent recurrence of an interval of approximately ten years between crises and, following a suggestion made by the astronomer, Sir William Herschel, in 1801, he sought to connect the rough periodicity of crises with the period of solar change as exhibited by the recurrence of maxima of sunspots. He did not suggest, as has been commonly asserted, that sunspots were the cause of crises, or of trade fluctuations, but pointed out that there was a close correspondence between the

[4] *Studies in Deductive Logic*, Preface, p. xii.

average period of recurrence of sunspot maxima and the average interval between crises. It was reasonable to suppose that the increase and decrease of the sun's spotted area indicated changes of the sun's condition which might affect solar radiation. This would cause variation of the weather in a corresponding cycle of years, which would affect harvests, and these in their turn the demand for industrial products, and so the activity of trade and industry.

Jevons' writings on the theory of economics are: the paper already mentioned which was read before the British Association, the *Theory of Political Economy;* and the posthumous *Principles of Economics.* The primer of political economy, written with unusual clearness and simplicity, is very elementary, contains no mathematical treatment, and to a large extent follows the conventional teaching of the time. Neither in his college teaching nor in his textbooks did Jevons attempt to force on students his (then) peculiar views.

The *Theory of Political Economy* has two distinctive features: the psychological basis and the mathematical method, which are not necessarily associated. The former he derived mainly from Bentham, and developed his calculus of pleasure and pain. These he regarded as opposite states of feeling caused by sensations of every kind and capable of neutralizing one another. Anything which affords pleasure or wards off pain at the time of consumption or use has actual utility. A pleasant or painful state of feeling is also produced by *anticipation* of the use or consumption of anything, whether in fact consumption of the particular thing or activity would or would not have the anticipated results. Anything to which a person attributes the power of increasing his pleasure or warding off pain when its consumption is anticipated, has for him *prospective* utility. It is in order to gain prospective utility that production and exchange are carried on. The rate of gain of prospective utility in time may be balanced against the present disutility, or painfulness, of the effort of work, as in the theory of labor (Chapter V) in which the psychological reaction leading to the cessation of labor is analyzed.

His mathematical method consisted, as is well known, in applying the methods of the differential calculus, or at least its ideas and symbols, to elucidate man's activities in satisfying his wants. In the theory of exchange, quantity of commodity is taken as the independent variable, in the theories of production and distribution quantities of labor and capital alternatively. Perhaps the most important contribution which Jevons made to economic thought in the seventies of the last century was the clear distinction he made between total utility and its first derivative, which he called final degree of utility.

It is perhaps unfortunate that he set himself the task of stating his

theory in mathematical garb and in popular terms in the same book. The whole of his theory, and more than he ever found time to write out, could have been expressed in mathematical symbols in a few pages. A more popular exposition would have followed naturally, but separately. As it is, mathematicians have always been irritated by repetitions and superfluous explanations, and non-mathematical readers have been puzzled or frightened off on first perusal by the alarming display of mathematical symbols. Apparently this mistake was due to his sense of the urgency of converting contemporary economists, the great majority of whom were non-mathematical, to acceptance of his quantitative method and its clear definitions.

In the application of mathematics to economics, Jevons was, of course, anticipated by Cournot, though at the time he was unaware of this; but the latter had no conception of marginal utility or of the equation of exchange, having confined himself to market demand and supply, monopolies, rent, etc. It remained for Walras on the Continent to develop independently the mathematical conceptions of utility, without the hedonic basis. The calculus of pleasure and pain, and the development of the idea of marginal utility from that, had been anticipated by only one man, Gossen. The latter developed the foundations of the theory elaborately, but used almost exclusively straight line functions, and failed to discover anything which could rank as a contribution to the theory of distribution.

Jevons has been criticized by Marshall and others for over-emphasizing demand as the determinant of value. It may be admitted that the book contains more matter relating to demand than to supply as determining price. This was clearly due to the author's desire to correct the prevailing doctrines which practically ignored demand and its changes, except in special cases. Careful reading shows that he was fully aware of the effects of differences or changes in the supply of the factors of production in the causation of price.

Questions of greater interest are how far Jevons understood the theory of distribution as subsequently generalized by Marshall, and how far his book contributed to Marshall's mathematical development of the theory. That Jevons had a good appreciation of the general theory of marginal equilibrium is clear from several passages, some of which it may be of interest to quote. In the "Theory of Labour,"[5] for instance, we find the following: "Thus we have proved that commodities will exchange in any market in the ratio of the quantities produced by the same quantity of labour. But as the increment of labour considered is always the final one, our equation also expresses the truth,

[5] Second edition, pp. 203–204; fourth edition, p. 187.

that *articles will exchange in quantities inversely as the costs of production of the most costly portions, i.e., the last portions added.* This result will prove of great importance in the theory of Rent.

"Let it be observed that, in uniting the theories of exchange and production, a complicated double adjustment takes place in the quantities of commodity involved. Each party adjusts not only its consumption of articles in accordance with their ratio of exchange, but it also adjusts its production of them."

Again a little further on[6] we read: "The quantities of commodity given or received in exchange, are directly proportional to the degrees of productiveness of labour applied to their production, and inversely proportional to the values and prices of those commodities and to their costs of production per unit, as well as to their final degrees of utility." The expression "degree of productiveness of labour" is clearly the same idea as marginal net product or marginal productivity of labour, because the first passage states that the increment of labour considered is always the final, that is, marginal, one.

In the "Theory of Rent" we read of equality of the "productiveness of labour, as regards the last increment of labour applied" when applied to different pieces of land; and "we may say, then, that whenever a labourer or body of labourers distribute their labour over pieces of land with perfect economy, the final ratios of produce to labour will be equal."[7]

In the succeeding paragraph and subsequently, however, the same quantity is termed "final rate of production." There is an important paragraph in the chapter "Theory of Capital," at the end of the section on the Tendency of Profits to a Minimum, in which the author makes very clear the relations between wages, rent, and interest, and the difference between the "whole yield" and "the final rate of yield." There is no suggestion that I can find, however, of that general view of distribution which regards the rate of rent of land as the marginal productivity of land varied in respect of a combination of the other factors of production (and so on, as regards each of the factors of production considered as variable in relation to the others, thus unifying the theory of the demand for the factors of production); though in the latter part of the preface to the second edition he remarks on the parallelism of the theories of rent, wages, and interest.

Of the numerous additions made in the second edition, perhaps the most important after the long new preface are the sections on the dimensions of economic quantities. The late Professor Allyn A. Young wrote

[6] Second ed., p. 209; 4th edition, p. 192.

[7] Second edition, p. 235, fourth edition, pp. 216–217.

in regard to this innovation:[8] "Here Jevons illustrates to perfection his extraordinary power of raising new questions of the utmost significance without satisfactorily solving them. In some respects his treatment is perfunctory and superfluous, but suggestive and tantalizingly incomplete, and not altogether accurate." I cannot agree that the treatment is either perfunctory or superfluous. The theory of dimensions is indispensable for clear thinking in the theory of economics, and it would be an advantage if more attention were paid to it now.

The *Principles of Economics* was intended to be a general treatise on economics suitable for the non-mathematical reader. It is written in a clear, easy style with a wealth of examples, many of great interest. Judging from the few chapters which were written, however, it does not seem likely that the book would have made any considerable contribution to the further development of economic theory. The variety of Jevons' activities when resident in London, and the poor health of his last few years, seem to have prevented that continuous mental concentration by which alone the pure theory of economics could be fully developed.

It is difficult to estimate the extent of the contribution which Jevons made to the development of modern economic theory. He was certainly the first to win acceptance for and popularize the theory of marginal equilibrium. There is evidence that Marshall gained ideas from the paper read before the British Association, published in 1866, and from the *Theory of Political Economy*, which found their place in his *Principles of Economics* (first published in 1890). On the other hand, Marshall, his interest in economics once aroused, would almost certainly himself have developed sooner or later the equation of exchange and the theory of distribution as they appear in his *Principles*, even if he had remained ignorant of the writings of Jevons, Gossen, Walras, and others.

Perhaps Jevons' major contributions to economic theory were his exposure of the fallacies of the Ricardo-Mill School, his establishment of the theory of utility as the basis of demand, the theory of exchange and its implications, and the true theory of wages. He did much to popularize the study of pure theory and to gain recognition of its mathematical basis. He accomplished the objects which he had closely at heart, and was perfectly conscious of the fact that he was leaving the greater part of the field of economic theory for others to work out. In the preface to the second edition of the *Theory* he wrote: "I must beg the reader to bear in mind that this book was never put forward as containing a systematic view of economics. It treats only of the

[8] *American Economic Review*, II (1912), 576.

theory, and is but an elementary sketch of elementary principles. The working out of a complete system based on these lines must be a matter of time and labour, and I know not when, if ever, I shall be able to attempt it." The fact is that economic theory was but one of his interests, and this has to be remembered if his few writings on the subject are contrasted with the output of professors of economics of later times. Probably he gave only a very small portion of his working time to the pure theory of economics. Several projects, such as the preparation of an annotated edition of the *Wealth of Nations*, were commenced, but never finished. Studies in logic, in statistics, banking, currency, and social reform, seem to have had the first claim on his time.

London, England

DÉVELOPPEMENTS RÉCENTS DE LA TECHNIQUE STATISTIQUE

PAR G. DARMOIS

Introduction.—Il faut redire avec W. A. Shewhart, qui a inauguré dans ce journal la série des rapports sur la statistique, qu'il s'agit en vérité d'un très vaste sujet.

Car son objet est la mise en ordre des observations, la recherche des régularités empiriques, suivie de la recherche des régularités logiques déduites d'une théorie, et enfin l'étude des raisons que nous avons de croire à la solidité de la construction ainsi obtenue, à sa qualité comme charpente des faits observés et de ceux que nous observerons plus tard. Or, tout cela n'est pas autre chose que le travail scientifique le plus général. Pourtant, dans les différentes figures que peut prendre ce problème, la statistique n'intervient pas également. Je ne veux pas dire seulement qu'il y ait, malgré tout, differents types de lois scientifiques, au moins pour la conception que nous en avons. Car le type de la loi fonctionnelle et le type de la loi statistique ne sont qu'une question d'échelle. Je pense plutôt à la différence entre l'observation en gros et en détail.

Le problème de la distribution des étoiles dans notre galaxie, de la forme et des mouvements de ce groupe stellaire est pour nous très profondément statistique, et de ce fait très difficile à résoudre. Nous voyons les étoiles en détail. Le problème serait infiniment simplifié si nous apercevions sur la rétine d'un observateur éloigné l'image de notre monde stellaire. Il y a dans la *forme* des lois, quelque chose de conventionnel dont notre esprit peut s'affranchir, mais il y a, du moins à un moment donné, *des modes d'observation* dont nous ne pouvons nous affranchir. Nous voyons les choses en gros pour les gaz, nous sommes bien forcés de les voir en détail pour les étoiles, bien que notre désir soit de les voir en gros.

On peut dire que la statistique intervient dans un problème quand le phénomène *élémentaire* est à l'échelle des observations. Un physicien qui mesure une longueur, fût elle une longueur d'onde, ne fait pas de statistique. Mais s'il observe un mouvement Brownien, s'il dénombre des émissions de particules par un noyau, il fait de la statistique.

Et si la statistique est appelée à jouer un rôle très grand, c'est que la plupart des problèmes qui intéressent les hommes sont normalement à leur échelle, et de plus, les anciens problèmes, dont on possédait une solution macroscopique assez fouillée et qu'on pouvait croire profonde, ont pris une forme nouvelle par le perfectionnement des moyens d'observation. Nous sommes entrés au cœur de la variabilité.

Par ailleurs, et même dans les cas ou nous avons de sérieuses raisons de penser qu'à l'échelle des observations, la grandeur est peu fluctuante, ses manifestations, qui sont le seul moyen pour nous de l'atteindre, se produisent dans des circonstances un peu variables qui l'affectent d'erreurs.

Ainsi, même quand les lois sont conçues sous la forme tranchante de liaisons fonctionnelles, elles apparaissent aux observations avec une certaine épaisseur, si j'ose dire.

Et c'est encore le rôle de la statistique de trouver la vraie position de la loi. Mais, si important qu'il soit dans ce cas, il n'est que peu de chose auprès de celui qu'elle joue dans le cas général, qui contient le précédent et où les lois, pour reprendre l'expression précédente, ont de l'épaisseur dans plusieurs dimensions.

Quand je dis que ces lois existent, je devrais dire qu'on espère les trouver, et que les observations devront encore en révèler la stabilité et l'homogénéité. Car nous serons trop souvent, je l'ai déjà dit, au cœur du phénomène élémentaire, et notre enquête, qui ne porte que sur des morceaux d'espace ou de temps, peut ne nous révèler, si nous manquons d'habileté ou de chance, que des traits locaux ou non représéntatifs.

La structure des lois éssayées.—On peut dire d'une manière générale qu'une observation faite et qui spécifie l'état de n grandeurs sera représentable par un point d'un espace à n dimensions. Cet espace sera rapporté à un système de coordonnées ordinaires si nous avons affaire à des grandeurs continues, il sera simplement découpé en cellules si les grandeurs elles mêmes sont seulement découpées en états.

Alors l'ensemble des observations nous donne un nuage de points. C'est la structure de ce nuage qui suit une loi, s'il en existe une, et c'est ce nuage qui doit nous aider à préciser nos idées et à découvrir une loi.

Cette loi sera conçue, soit comme la densité d'un nuage continu dont celui des observations serait une épreuve, soit comme une distribution de fréquences attribuées à chaque cellule de l'espace.

Bien entendu, la variété est infinie dans les possibilités de notre esprit, et seule une connaissance vivante de la question à l'étude peut suggérer soit une ébauche à perfectionner, soit une forme précise. On peut dire que beaucoup de résultats analytiques ont été établis en supposant que le nuage de points, à une ou plusieurs dimensions, suit une loi de Laplace-Gauss. Il y a là une sorte de tyrannie qui étonne souvent, mais qui se justifie par l'importance tout à fait exceptionnelle de cette loi.

Comme il est bien connu, une des raisons qui expliquent cette apparition constante est dans une propriété très générale. Si on ajoute un grand nombre de variables aléatoires indépendantes suivant des lois quelconques, la variable résultante suit une loi voisine de la loi de

Gauss, à condition que chacune des variables composante soit petite par rapport à l'écart type de la somme (cette façon de parler un peu vague remplaçant un énoncé précis).

Dans ces conditions, il suffit de peu de chose pour que la variable étudiée relève raisonnablement d'une loi de Gauss. Si elle nous apparait comme une résultante de nombreuses causes comparables, sans analyse approfondie de ces causes, nous pourrons supposer une structure de Gauss, à condition que ces causes soient indépendantes. Et ce théorème reste vrai si les composantes sont des vecteurs aléatoires; nous avons ainsi des lois de Gauss à plusieurs variables.

Il est important de remarquer que l'indépendance des composantes n'est pas nécéssaire, et que les résultats sont valables sous des conditions beaucoup plus larges, qu'on peut indiquer de la manière suivante:

Une variable aléatoire conçue comme une somme apparait généralement comme le terme d'un développement linéaire, ou les composantes s'ajoutent dans un certain ordre. On peut donc parler de composantes successives, de composantes éloignées.

Dans ces conditions, et généralisant des résultats de Markoff sur les sommes de variables dépendantes, Serge Bernstein[1] a montré que si la liaison s'affaiblissait suffisamment avec la distance, la somme des variables composantes continuait à suivre une loi de Gauss. Or, il arrivera souvent que les termes successifs nous apparaitront comme dépendants, mais que cette dépendance disparaitra au bout de quelque termes. On pourra donc, dans des cas beaucoup plus généraux, escompter le même résultat.

On voit en particulier que la théorie des erreurs dépendantes introduira encore pour l'erreur totale, des lois de Gauss, mais il convient de remarquer que l'étude de la précision des mesures devra tenir compte du fait que les séries experimentales sont liées.

Il résulte également de ces recherches une extension évidente du champ d'application des lois qui généralisent la loi de Gauss. Ces cas interviennent quand les causes aléatoires produisent sur la grandeur observée un effet dependant de la grandeur elle-même. On voit qu'on pourra supposer les causes aléatoires dépendantes, sous certaines conditions, et que les distributions garderont la même forme.

Toujours dans cet ordre d'idées, on sait quel inappréciable matériel les travaux de Karl Pearson ont fournit à ceux qui cherchent à résoudre, comme l'a dit R. A. Fisher, ce problème préliminaire de la spécification.

Il est donc bon de rappeler à ce propos que Polya,[2] dans une élégante étude qui illustre les résultats de Markoff et S. Bernstein, a montré de

[1] S. Bernstein, *Math. Annalen.* T. XCVII, 1927.

[2] G. Polya, *Annales de l'Institut Henri Poincaré*, 1930.

plus que parmi les cas limites fournis par la somme de variables dépendantes, se trouvaient deux des répartitions de Karl Pearson, et précisément celles qui ont connu le champ le plus étendu d'applications.

Ces résultats nous paraissent essentiels sil'on veut voir, dans le problème de la spécification, non pas un simple choix plus ou moins heureux dans l'arsenal existant des courbes de fréquence, mais plutôt ce qu'il doit être en réalité: le premier pas, fait par quelqu'un qui connait personnellement son terrain, sur la voie d'une théorie qui cherche à se préciser.

Cas ou l'on possède une forme analytique pour la structure.—On voit que malgré tout, ce cas peut se présenter assez fréquemment. Il faut alors ajuster au nuage experimental cette structure analytique. Il faut redire ici la valeur exceptionnelle de la contribution apportée à cette question par R. A. Fisher dans un ensemble de travaux où la théorie de l'estimation a pris une forme si précise, où des idées dont la fécondité n'a pas cessé de se révéler ont transformé et orienté le problème.

On sait que la base de ces travaux est la notion de "likelihood" et que dans l'hypothèse d'une structure fixée, la méthode dite de "maximum likelihood" fournit les estimations les plus précises et permet d'utiliser au mieux les renseignements fournis par l'observation.

On peut signaler ici qu'au point de vue pratique, qui est fort important, cette méthode peut présenter de sérieuses difficultés. R. A. Fisher a indiqué lui-même, dans un de ses mémoires les plus suggestifs[3] une méthode qui permet d'atteindre un résultat équivalent, à partir d'une solution moins précise.

Comme la méthode des moments est la plus généralement adoptée, il est intéressant de signaler ici un travail de R. S. Koshal[4] sur l'amélioration de cette méthode classique par l'emploi du procédé d'approximations de R. A. Fisher.

Dans un ordre d'idées voisin, nous indiquerons aussi les mises au point de la méthode des polynomes orthogonaux pour la représentation des observations. Un fascicule des *Annals of Mathematical Statistics* rassemble un travail de Ch. Jordan où l'on trouvera des tables complètes (novembre 1932).

Ces questions ont été également traitées, tout à fait à fond, d'une manière élégante et qui semble définitive dans un travail d'Aitken.[5]

De très importants résultats, applicables à la représentation des

[3] "Theory of Statistical Estimation," *Proceed. of the Cambridge Phil. Soc.* XXII, p. 700.

[4] *Journal of the Royal Statistical Society*, XLVI (1933), 303.

[5] "On the Graduation of Data by the Orthogonal Polynomials of Least Squares," *Proceed. of Royal Soc.* (Edinburgh, 1933). LIII, 54–78.

séries statistiques, ont été également obtenus par R. Frisch[6] et A. Guldberg. Depuis longtemps, R. Frisch avait publié des résultats sur le premier moment incomplet de la distribution binômiale.

Il a repris la question à la suite d'un travail d'Alf. Guldberg paru en 1931, et qui mettait en évidence l'avantage procuré par une équation différentielle (aux différences finies) vérifiée par la fonction de fréquences. Dans un travail de R. Frisch,[7] et dans une série de leçons faite par Alf. Guldberg[8] l'utilisation de cette équation différentielle permet d'obtenir des expressions simples pour les moments complets ou incomplets.

L'équation peut toujours être écrite en utilisant des caractéristiques de la distribution et fournit un critère simple pour reconnaître rapidement si une série statistique peut être représentée par une distribution théorique choisie.

Les critères dits locaux sont déduits de cette équation elle-même. Des critères globaux, applicables à l'ensemble de la distribution peuvent être obtenus à l'aide des moments. Ces résultats peuvent s'étendre à deux ou plusieurs variables comme l'a montré Alf. Guldberg.[9]

Le troisième problème.—On a précisé numériquement la valeur des paramètres d'une distribution ajustée aux observations. Les observations permettront toujours de tels calculs. Mais il reste à se demander si ces observations sont d'accord avec les résultats, autrement dit si l'ajustement est satisfaisant. Ce problème, auquel Karl Pearson et son école, directe ou indirecte, ont consacré un magnifique et fructueux effort, comporte plusieurs parties.

Il faut d'abord savoir comment se comportent, pour une population de structure supposée fixe, les caracteristiques du nuage des observations. La solution de ces questions est d'ailleurs essentielle pour le problème de l'estimation.

On a obtenu sur ce premier point des résultats très étendus. La structure des nuages expérimentaux déduits d'une loi de Gauss a été étudiée pour un nombre quelconque de variables, cela principalement par R. A. Fisher et ses élèves, et des formes analytiques précises de lois de distributions ont été obtenues.

D'autre part, des méthodes régulières très importantes permettront d'étudier, pour une répartition quelconque, les lois des probabilités des moments expérimentaux relatifs au nuage.[10]

La deuxième question, qui utilisera les résultats de ces premières re-

[6] *Skandinavisk Aktuaritidskrift*, p. 267.
[7] *Metron*, x, 36.
[8] *Annales de l'Institut Henri Poincaré*, 1933.
[9] *Comptes Rendus de l'Académie des Sciences de Paris* xcvi, 1934.
[10] N-St. Georgesco, *Biometrika* (1932), p. 65.

cherches, est la suivante. Comment jugera-t-on si les hypothèses qui ont servi de point de départ à l'estimation resteront d'accord avec l'ensemble des résultats expérimentaux? C'est à quoi répondent les travaux de Karl Pearson, et pour le cas d'une distribution de fréquences, l'emploi comme élément essentiel de χ^2, généralisant comme l'a montré R. A. Fisher, le rapport de Lexis.

La loi de probabilité de χ^2, quand il s'agit de différences entre l'observation et l'ajustement, a été précisée par R. A. Fisher, et plus récemment par E. S. Pearson et Neyman. Ces deux auteurs ont repris à la base la question générale de l'accord des hypothèses et de l'observation. Les travaux déjà analysés par W. A. Shewhart ont été suivis par un mémoire remarquable et très suggestif,[11] "On the problems of the most efficient tests of statistical hypothesis."

Comme le rappellent les auteurs, il s'agit d'un très ancien et très passionnant problème. Depuis le mémoire célèbre de Thomas Bayes (*Philos. Trans.* Vol. 53, 1763) qu'il intitulait si modestement "An Essay towards Solving a Problem in the Doctrine of Chances" et qui ne fut communiqué qu'après sa mort, ce problème a exercé, et n'a pas fini d'exercer la sagacité d'un grand nombre de chercheurs. Il a fait réfléchir et discuter, parfois avec quelque vivacité, un très grand nombre de savants.

On adopte en général, dans ce problème, l'attitude suivante: Ayant choisi un caractère du groupe des observations obtenues, on cherche quelle serait, en conséquence de l'hypothèse choisie, la probabilité pour que ce caractère ait la valeur observée. Si l'on trouve que cette probabilité est très petite, on tend à suspecter l'hypothèse. Sinon, on la conserve. Quand cette méthode est employée sans précaution, elle est éminemment simpliste et mérite les critiques que lui a faites Joseph Bertrand. La base n'est pas sérieuse. Pourtant, si le choix du caractère est fait assez habilement, par quelqu'un qui comprend bien la question, si comme le dit Borel, le caractère adopté est en quelque sorte remarquable, la méthode peut donner, et donne en fait, de bons résultats. Une étude systématique est alors faite du point de vue suivant: Le test doit avoir pour but une règle de conduite à suivre vis à vis de l'hypothèse. Si le caractère se trouve dans une certaine région dite critique, l'hypothèse sera suspectée, si le caractère se trouve en dehors de la région critique, l'hypothèse sera conservée.

Il est clair que si nous parvenons à montrer: (1°) qu'une telle règle ne rejette que rarement une hypothèse vraie; (2°) qu'elle ne garde que rarement l'hypothèse quand elle est fausse nous aurons à notre disposition un test efficace.

[11] *Phil. Trans. of the Royal Society of London*, A. CCXXXI, 289.

Il est alors intéressant de se demander s'il est possible de construire un test qui soit le plus efficace. Cela voudra dire que le premier risque étant fixé à une valeur ϵ, le second risque η soit réduit à son minimum.

Quant à la valeur absolue de ces deux risques, c'est évidemment une affaire d'appréciation. Comme le disent à peu près les auteurs, lequel est le plus dangereux, d'emprisonner une idée vraie, ou de laisser courir une idée fausse?

Si nous nous bornons au cas simple où deux hypothèses seulement tiennent le champ, H_0 étant l'hypothèse que nous présentons, H_1 l'autre hypothèse possible, la région critique, celle où doit se trouver le caractère pour que l'hypothèse soit suspectée fournit une probabilité ϵ. C'est le premier risque calculé dans l'hypothèse que H_0 est vraie. Il y a évidemment une infinité de régions critiques qui donnent la même valeur de ϵ.

Existe-t-il une de ces régions pour laquelle la probabilité relative à l'autre hypothèse ait la valeur la plus grande possible $1-\eta$, η sera alors le second risque réduit au minimum.

On voit qu'il s'agit d'un problème de calcul des variations. Il imposera une condition à la frontière de la région critique.

Bien entendu, le problème dans le cas où il existe plusieurs hypothèses du type H_i est beaucoup plus difficile.

Une solution est proposée qui réussit dans certaines conditions. Bien que cette dernière question ne soit pas résolue d'une manière absolument satisfaisante, on doit considérer comme très intéressants les résultats obtenus.

Il est remarquable qu'étudiant dans cet ordre d'idées l'hypothèse d'une moyenne donnée, avec un écart type inconnu dans une population normale, les auteurs aient retrouvé comme le meilleur test celui qu'on applique précisement à ce problème depuis le mémoire de "Student."

Si ce résultat fait apparaître l'habileté, le bon sens, et comme le disent les auteurs "l'heureuse intuition" qui a présidé au choix du caractère adopté, il ne faudrait pas en conclure que cette intuition puisse suffire dans des cas plus complexes, où les principes établis auront leur application naturelle.

Signalons enfin le rôle joué par la notion de *likelihood* dans ces questions. Il arrive qu'il n'existe pas de région critique qui soit la meilleure, mais on peut toujours en obtenir une qui soit satisfaisante, en la limitant par un contour où la vraisemblance ait une valeur constante.

Essai d'interprétation des distributions obtenues.—Le fait qu'on a obtenu une concordance satisfaisante avec l'hypothèse choisie, et que celle-ci soit maintenue, ne peut pas être considéré comme le terme absolu de la recherche. Il reste toujours, on le sait bien, quelque chose à

comprendre, mais dans ce cas, le désir de comprendre peut être particulièrement justifié. Car en général, même si des considérations théoriques ont conduit à la forme adoptée comme structure de la population, ces réflexions peuvent ne pas avoir une forme très précise.

Il est évident par exemple que les lois de Gauss trouvées par les biométriciens sont déjà un résultat très satisfaisant, une assez profonde description. Ce sont des formes, qu'il était naturel d'essayer, avec des idées même assez vagues sur l'origine des grandeurs mesurées. Il n'en est pas moins vrai qu'il serait important de dissocier ces variables aléatoires en diverses composantes considérées comme plus simples. On peut considerer cette question comme résolue de manière satisfaisante par divers mémoires, parmi lesquels ceux de Karl Pearson et R. A. Fisher, qui ont montré que les lois de Mendel associées à l'action cumulative des différents facteurs mendéliens, expliquent jusqu'aux valeurs numériques les résultats observées. Ainsi, nous pourrons nous montrer plus satisfaits si, au lieu de décrire simplement la forme du nuage des observations par un certain nombre de moments, nous parvenons en quelque sorte à reconstruire cette forme à l'aide de composantes considerées comme plus simples, en particulier indépendantes entre elles.

Les liaisons observées résultent alors simplement du fait que les variables indépendantes sont communes, en tout ou en partie, aux variables observées.

C'est ce type d'explication qui est à la base de la théorie de Spearman du facteur mental principal, et des facteurs spécifiques. On voit qu'on peut chercher à représenter les variables observées x_i qu'on peut supposer réduites à leur centre de gravité et divisées par leur écarts types, par des formules du type

$$x_i = \sum a_{i\gamma}\gamma_\gamma$$

où le nombre des composantes γ_γ serait *a priori* quelconque. Il est bien clair que si l'on suppose normal le nuage continu dont les x_i sont extraits, la forme quadratique $E(u_i x_i)^2$ qui donne les écarts types et les coëfficients de corrélation devra être égale à la somme des carrés des formes linéaires

$$a_{ij}u_i = v_j,$$

et l'on aura ramené au nuage normal des variables indépendantes γ le nuage expérimental.

Le problème est toujours possible d'une infinité de manières, mais il parait intéressant surtout dans le cas ou les γ sont en même nombre que les x.

On voit en effet qu'il se ramène à la réduction d'une forme quadratique.

Il a été envisagé d'une façon générale par R. Frisch[12] et a été repris récemment par H. Hotelling[13] qui donne le moyen pratique de réduire la forme quadratique

$$E(u_i x_i)^2.$$

Les composantes principales apparaissent dans l'ordre de grandeur décroissant des axes principaux de l'ellipsoïde dont cette forme quadratique est la forme tangentielle.

La première composante est la distance au plan principal perpendiculaire au plus grand axe du nuage.

On voit que si les contributions des dernières composantes peuvent être jugées assez faibles, il devient permis de les attribuer au hasard et de considérer la liaison mise en évidence comme analysée suffisamment pour les composantes qui restent.

Une méthode est donnée pour juger si les différences des racines sont significatives et si une composante peut être négligée. Enfin, la fin du travail essaie de résoudre la question suivante: peut-on juger sur les résultats obtenus une hypothèse proposée sur le nombre des composantes principales?

La signification des lois statistiques.—Le problème fondamental de la recherche des lois statistiques a été discuté récemment par Ragnar Frisch, dans les dernières leçons d'une série traitant également d'autres problèmes. Ces leçons ont été données à l'Institut Henri Poincaré en 1933, et doivent paraître sous peu de temps dans les *Annales de l'Institut Henri Poincaré*. Nous voulons seulement indiquer ici, avec quelques résultats importants, l'esprit de ces recherches.

Il s'agit d'abord du problème de l'invariance des lois statistiques, dont l'auteur s'occupe depuis plusieurs années, et en liaison avec ce premier problème, du fondement philosophique des lois d'observation, sur lequel R. Frisch exprime des idées très personnelles, et dont la hardiesse ne peut manquer de susciter une discussion féconde.

Pour la première question, l'auteur a démontré qu'il n'existe pas de regression linéaire invariante dont les coefficients soient des fonctions continues des moments, fonctions pourvues de dérivées continues.

En outre, il a montré qu'on peut toujours, par une tranformation convenablement choisie effectuée sur les variables, obtenir une corrélation aussi étroite qu'on le désire, et une pente quelconque des lignes

[12] "Correlation and Scatter in Statistical Variables," *Nordic Statistical Journal*, III, 56. "Statistical Correlation and the Theory of Cluster Types,' *Jour. of Amer. Statistical Assoc.* (1931), p. 375.

[13] "Analysis of a Complex of Statistical Variables into Principal Components," *Journal of Educational Psychology*, September–October, 1933.

de régression. Ce résultat, complété par le fait que cette transformation peut être construite à l'avance, sans que soit connue la distribution statistique véritable des variables à transformer, est totalement explicité en formules pour le cas de deux variables, mais il est, sans doute, général.

Ces conceptions conduisent l'auteur à se faire une représentation de la science comme une sorte d'ample recherche d'une transformation singulière de l'univers.

En particulier, pour les recherches biologiques et le problème de l'évolution, il suggère une idée de ce que pourraient être en réalité les régularités que nous croyons découvrir.

Ces régularités qui sont, sans doute, indispensables à la survivance de toutes les formes de vie qui nous sont connues, et en particulier à la survivance de notre espèce, ne seraient que le résultat d'une évolution singulière des organes des sens des différentes espèces.

Par cette évolution, les faits réels ne nous parviendraient, si l'on ose dire, que filtrés et choisis, nous n' observerions qu'un ensemble de variables transformées où apparaissent précisément des régularités.

On voit que ces vues conduiraient à considérer l'évolution de la vie comme un immense processus d'adaptation, par lequel nous deviendrions de plus en plus aptes à transformer les chocs qui nous viennent d'un univers essentiellement chaotique en un fantôme "d'ordre et beauté" qui nous donne l'impression de permanence et de régularité.

On voit ici que la statistique, dont l'étendue nous avait déjà paru si imposante, et qui vient de s'annexer récemment de vastes champs de la physique, poursuit ses conquêtes en direction de la philosophie.

Sur l'emploi du coefficient de corrélation.—Nous voudrions signaler à part un résultat très simple et frappant, obtenu par M. Fréchet dans une question importante: l'emploi du coefficient de corrélation pour dégager des relations et mesurer l'étroitesse des dépendances.

Etant donné un nuage dans le plan, ce qu'on appelle d'ordinaire le coefficient de corrélation r peut être mis sous forme d'un produit de deux facteurs:

1°. Le soi-disant coefficient de corrélation obtenu en associant à chaque valeur de x la moyenne des valeurs de y comptée avec le poids correspondant soit ρ ce premier facteur.

2°. Le rapport de corrélation formé par la moyenne des fluctuations des y dans chaque colonne x_i, divisée par la fluctuation totale des y.

Le premier coefficient ne dépend que de la courbe des moyennes et non de la dispersion; le deuxième n'est pas affecté au contraire quand on déplace les distributions partielles de manière à déformer la courbe des moyennes.

De sorte que le facteur ρ est étranger à l'étroitesse de la dépendance.

D'autre part, comme l'indique M. Fréchet,[14] le rapport de corrélation peut être voisin de 1 pour deux raisons, soit parce que les nombreux points y correspondant à x sont voisins de la courbe moyenne, soit parce qu'il y a un petit nombre de valeurs de y pour chaque valeur de x. Ce sera précisément le cas dans les séries temporelles. De sorte qu'il sera tout à fait illégitime de déduire d'un indice de corrélation une conclusion sur la précision de la relation qui détermine y connaissant x.[15]

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[14] *Revue de l'Institut International de Statistique* (La Haye), 1 (Janvier 1934), p. 16.

[15] Je signale ici la publication par J. O. Irwin du troisième de ses si utiles et intéressants rapports intitulés "Recent Advances in Mathematical Statistics, 1932," où le lecteur pourra trouver une bibliographie très étendue. *Journal of the Royal Statistical Society*, Vol. XCVII, Part I, 1934.

A CRITICAL EXAMINATION OF PROFESSOR PIGOU'S METHOD OF DERIVING DEMAND ELASTICITY

By R. G. D. Allen

Professor Pigou's well-known method of estimating the elasticity of demand for a commodity from price-quantity observations over time,[1] has been critically considered, first by Professor Ferger, and more recently by Dr. Cassels.[2] As a result, a number of objections, modifications, and extensions of the method have been advanced, and this fact justifies an attempt to obtain a clear estimate of the usefulness and applicability of Professor Pigou's method.

It is proposed here to investigate quite generally the possibility of extracting significant information on supply or demand elasticity from the usual kind of price-quantity data and by means of a formula of the type suggested by Professor Pigou.

The two most important considerations that must be taken into account in this connection are:

1. The nature of the information obtained depends on the *assumptions* which it is permissible to make about the nature of the shifts both of demand and of supply.

2. *Errors of observation* are present in any actual price-quantity data.

It is maintained here that, in general, too little attention has been paid to the second and too much attention to the first of these considerations.[3] Observation errors seem to be of rather more importance than *small* divergencies in the assumptions made, and it is suggested that the latter can be included as a subsidiary case of the former.

Our conclusions are in good agreement with those presented by Professor Frisch in his monograph: *Pitfalls in the statistical construction of demand and supply curves* (1933). Professor Frisch shows, indeed, that statistical estimates of demand and supply elasticity, based on mathematical constructions, frequently can produce results which are meaningless. As an example, Professor Frisch considers in detail Dr. Leontief's method[4] and, as a result of Professor Frisch's criticisms, it is found that the practical value of the method is severely restricted. It is shown below that the same objections apply to Professor Pigou's method of deriving demand curves. The contention is, however, here

[1] *Economic Journal*, September 1930, pp. 384–400.

[2] Ferger, *Economic Journal*, March 1932, pp. 17–26. Cassels, *Economic Journal*, December 1933, pp. 575–585.

[3] This is certainly true of Dr. Cassels, but it should be noticed that Professor Ferger provides a brief treatment of observation errors (*op. cit.*, p. 20).

[4] *Weltwirtschaftliches Archiv*, July 1929.

made that Professor Pigou's method, generalized and used with care, can under certain circumstances provide significant information in cases where Dr. Leontief's method fails.

The fact that the information to be obtained from price-quantity observations depends in a *symmetrical* manner on the conditions of demand and supply of the commodity was recognized by Professor Ferger.[5] This symmetry was hidden in Professor Pigou's work by the special nature of the assumptions he made about the variation of demand conditions. In order to exhibit the symmetry, therefore, it is necessary to start with a scheme in which demand and supply conditions are on an equal footing, and to consider various sets of assumptions that can be made about the variation of demand and supply over time.

It is assumed, first, that the conditions of demand and supply can be represented by schedules connecting the quantity of the commodity demanded or supplied with the price ruling in a given interval of time.[6] The schedules can then be plotted on a diagram as curves referred to two perpendicular axes OP and OX, where prices are measured along OP and quantities along OX, a logarithmic scale being adopted in each case. The slope of the tangent referred to OP gives the elasticity of the schedule at any point. A schedule of constant elasticity, therefore, is represented by a straight line with slope equal to the value of the elasticity. We define a *regular shift* of such a straight line schedule over time as a uniform and *progressive shift* of the schedule (always to the

[5] *Op. cit.*, p. 24. See also Cassels, *op. cit.*, p. 581.

[6] There is little difficulty about the demand schedule. The static form can be taken in which demand is a function of the income distribution and of the prices of all commodities. The elasticity of demand is then the (standardized) partial derivative of demand with respect to the price of the commodity in question. The demand schedule is obtained from the demand function by giving a *hypothetical* set of values to the price of the commodity, other prices and the income distribution being fixed. This does not exclude variations in the other prices and in the income distribution *over time*. On the contrary, the demand schedule shifts over time, and the shift is determined by factors including just these variations (see Pigou, *op. cit.*, p. 388).

The time element becomes more important on the supply side and there is on the whole more difficulty about the supply schedule (see Ferger, *op. cit.*, p. 25). But the assumption of the existence of a static supply schedule seems fairly reasonable in the case of a simple consumers' good (sugar, for example, as used in the illustration below) and when the interval to which prices and quantities refer is not too short, a year, for example, instead of a week or a month. Notice that pure theory indicates that a perfectly competitive industry, using no "scarce" factors of production, expands and contracts under conditions of constant average cost *equals* supply price. A gently rising or falling supply schedule can be expected to represent cases which do not differ greatly from this theoretical case.

same side) without alteration of its slope. A schedule which has shifted regularly over a series of successive intervals of time is represented by a set of equidistant parallel straight lines.

The most convenient classification of possible assumptions about the nature of the shifts of demand and supply now seems to be the following:

I. Both the demand and the supply schedules are straight lines of constant elasticity with regular shifts over time.

II. (1) The demand schedule is a straight line with a regular shift while the supply schedule is of any form with an irregular shift. (2) The supply schedule is a straight line with a regular shift while the demand schedule is of any form with an irregular shift.

III. Both the demand and the supply schedules are of any form (linear or otherwise) with irregular shifts over time.

The price-quantity observations at any three successive intervals of time give three points, $A(p_1, x_1)$, $B(p_2, x_2)$, and $C(p_3, x_3)$, on the double logarithmic diagram. All p's are logarithms of observed prices and the x's logarithms of observed quantities. Let B' denote the mid-point of AC with co-ordinates

$$\left(\frac{p_1+p_3}{2}, \frac{x_1+x_3}{2}\right).$$

A unique set of three equidistant parallel straight lines can be drawn to pass through A, B and C respectively. Since the middle line must bisect AC, it coincides with BB', The other two lines are parallels through A and C. The slope of the set of lines (referred to OP) is the slope of BB'

$$= \frac{\frac{x_1+x_3}{2} - x_2}{\frac{p_1+p_3}{2} - p_2} = \frac{x_1+x_3-2x_2}{p_1+p_3-2p_2}$$

i.e., slope of parallel lines

$$(1) \qquad = \frac{(x_1-x_2)-(x_2-x_3)}{(p_1-p_2)-(p_2-p_3)}.$$

This corresponds to the formula

$$\frac{d_1-d_2}{c_1-c_2}$$

of Professor Pigou.[7]

[7] Note that the formula (1) gives the slope of the three lines referred to OP, quantities appearing in the numerator and prices in the denominator. The for-

The formula (1) fails in one, and only one, case. This is the case where the three points are co-linear with B as the mid-point of AC. The points B and B' then coincide and an infinite number of sets of equidistant parallel straight lines can be drawn through the three points. The slope of formula (1) is completely indeterminate.

Suppose that the triad of points A, B, and C, are located without errors of observation. Each point then corresponds exactly to the intersection of the appropriate demand and supply schedules. What does formula (1) represent in terms of these schedules? The answer depends on the nature of the assumptions about the demand and supply shifts. In case I, both the demand and supply schedules shift regularly over time. Hence A, B, and C, are points of intersection of pairs of straight lines of two equidistant parallel sets. The triad is thus co-linear and B is the mid-point of AC. In this case the construction breaks down and formula (1) is completely indeterminate.

In case II (1) , the demand shifts are regular but the supply shifts are irregular. The points A, B, and C, are not co-linear and formula (1) gives the slope of the unique set of three equidistant parallel lines through them. This set must coincide with the three demand schedules, i.e., *formula* (1) *gives the elasticity of demand.* In the same way, in case II (2), the set of three equidistant parallel lines through A, B, and C, is unique and coincides with the supply schedules, i.e. *formula* (1) *gives the elasticity of supply.*

In case III, neither the demand nor the supply shift is regular. A unique set of three equidistant parallel lines can be drawn through A, B, and C, but the set does not coincide with the demand or with the supply schedules. Formula (1) is determinate but, except by accident, quite meaningless in terms of demand and supply.

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mula connects at once with the elasticity expression $d\log x/d\log p$. As pointed out by Professor Ferger, (*op. cit.*, pp. 19–20), two slips occur in the original statement of the geometrical property by Professor Pigou. The sign of the expression is left indeterminate and the ratio is reversed. These slips are not, however, carried over into the application of the formula.

This confusion arises from the difference between the conventional *economic* method of drawing demand and supply curves and the conventional *mathematical* method. The former takes OX as the horizontal axis and the latter takes OP horizontal (p being the independent variable). The slope of a straight line is a mathematical concept conventially referred to the horizontal axis; but it must here be referred to the vertical axis if the economic convention is adopted. Notice that the slope in Dr. Cassels' article is referred to OX (and not to OP) and so measures the reciprocal of the elasticity.

No diagrams are actually drawn in the present discussion. They can be readily supplied by the reader, remembering that the slopes are all referred to OP however the diagrams are drawn.

The effect of observation errors in the triad ABC on the interpretation of formula (1) must now be considered.

In case I, B will diverge from B' by a small observation error. Formula (1) thus gives a definite result—the slope of BB'. But the result takes the form of an observation error divided by an observation error, and the actual indeterminate form of (1) is masked by the presence of observation errors. This is a very simple case of what Professor Frisch calls "fictitious determinateness created by random errors."[8]

In cases II (1) and (2), the point B' is subject to an error of the same order as the observation errors in A, B, and C. The slope of BB', given by formula (1), is therefore subject to error due to errors in the location of both B and B'. The effect of this is dependent on the distance of B from B' relative to the size of the original observation errors. If BB' is relatively large, formula (1) is a close approximation to the elasticity of demand or supply. If BB' is relatively small, the slope of BB' can take almost any value according to the direction of the errors, and formula (1) ceases to be even a rough approximation to the elasticity of demand or supply. In intermediate cases, formula (1) gives a more or less close approximation to the elasticity according as B is more or less distant from B'.

In terms of the demand and supply shifts, formula (1) gives a more or less close approximation to the elasticity of demand if the demand shift is regular and the supply shift more or less irregular. Hence, if the demand shift is regular and the supply shift *almost* regular, then (1) gives the demand elasticity if no observation errors are present, but this elasticity is completely masked by the presence of errors. For formula (1) to apply, with observation errors present, it is necessary that the supply shift should diverge noticeably from regularity. Similar remarks apply if the supply shift is regular and the demand shift more or less irregular.[9]

[8] Frisch, *op. cit.*, pp. 7–10.

[9] These results can obviously be interpreted in terms of the *correlation* between the demand and supply shifts. If the shifts are perfectly correlated, we have case III or case I according as the shifts are both irregular or both regular. Except in the latter case, when the formula is indeterminate and indicates nothing, it is seen that formula (1) indicates the *trend*, i.e., the regression of quantities upon price in the triad. If the correlation is large but not perfect, formula (1) gives an approximation to the trend; even if one shift is regular, formula (1) does not give its elasticity, for the other shift is almost regular also. Apart from stable schedules, it is only when the correlation is fairly small and one shift is regular that formula (1) gives an elasticity. (Compare these conclusions with Ferger, *op. cit.*, pp. 22–23.) Professor Pigou's method is therefore subject to very much the same objections as that of Dr. Leontief. It is indicated below, however, that the objections do not impose so serious limitations on the applicability as in the case of Dr. Leontief's method.

In case III, the formula (1) is in any case meaningless in terms of the demand and supply schedules, and the presence of observation errors does not affect the position.

It is now a simple matter to allow for the effect of a *small* divergence from the assumption of regular shifts in cases I and II. In case II (1), suppose that the demand shift is almost, but not exactly, regular. If $1+\lambda:1$ (λ small) is the ratio of the shifts from the first to the second to the third positions, then the elasticity of demand is obtained as the slope of the line joining B to the point dividing AC in the ratio $1+\lambda:1$. (Observation errors are assumed to be absent.) But formula (1) is the slope of the line joining B to the mid-point of AC. The effect of using formula (1) to express the demand elasticity in this case is thus the same as the effect of a small observation error in B' masking exactly regular shifts. Formula (1) is still a close approximation to the demand elasticity except when the distance BB' is not large, i.e., except when the supply shift is, like the demand shift, almost regular. Similar remarks apply to the case II (2), and there is no further difficulty in combining errors of observation and small divergences from regularity of shift.

If there is a *considerable* divergence from the assumption of regular shifts in demand for supply, the case III is obtained at once and the formula ceases to have any meaning whatever in terms of demand or supply elasticity.

The case considered by Professor Pigou is that of a regular demand shift combined with any form of a supply shift. These special assumptions restrict him to the cases I and II (1). Allowing for *small* observation errors and for *small* divergences from regularity of demand shifts, it follows that Professor Pigou's formula only approximates closely to the elasticity of demand when BB' is reasonably large, i.e., when the supply shift is reasonably irregular. If the supply shift, as well as the demand shift, is actually regular, then the presence of observation errors gives rise to the case of fictitious determinateness. Further, if the supply shift approximates at all to regularity, the approximation of Professor Pigou's formula to demand elasticity ceases to be a close one.

Without *a priori* evidence about the nature of the demand and supply shifts, it is clearly a restriction to take Professor Pigou's unsymmetrical assumption. But the method can be generalized, on the lines indicated above, to allow for any possible assumption about the regularity or otherwise of either the demand or the supply shift. A convenient and simple practical method is the following:

A sequence of observed price-quantity triads, A, B, and C, at successive intervals of time provides the data. For each triad, the slope

of the set of three equidistant parallel lines through the points is obtained by formula (1). This is also the slope of the line joining B to B', the mid-point of AC. Each triad is then examined from the point of view of probable shifts in demand and supply and both internal and external evidence can be used here. As regards internal evidence, we can proceed:

1. Triads are rejected for which BB' is small relative to the general average and to the size of $\frac{1}{2}\ AC$. These triads probably correspond to the case of regular demand and supply shifts and the slope of BB' exhibits fictitious determinateness.

2. If evidence suggestions that demand and supply are fairly inelastic for the commodity,[10] triads are rejected for which the slope of BB' has a large *numerical* value (whether positive or negative). It is taken that this represents a case of irregular shifts in both demand and supply.

3. The remaining triads are examined critically with respect to the probable shifts in demand and supply. The relative positions of the points A, B, and C, provide the internal evidence, i.e., the distances and slopes of AB' and BB'. Some triads are rejected, some give estimates of demand elasticity (demand shifts regular) and others estimates of supply elasticity (supply shifts regular).

If sufficient triads remain giving estimates of the demand elasticity and of the supply elasticity, by means of formula (1), then Professor Pigou's method of taking the median and quartile deviation can be used to give final estimates of both elasticities and to indicate, to some extent, the significance of the estimates.

We are now in a position to compare the method of Professor Pigou, when generalized and scrutinized in practice on the lines indicated above, with more complicated methods of which that of Dr. Leontief may be taken as typical. Both methods involve the risk of obtaining a result which either displays fictitious determinateness due to observation errors or is meaningless in terms of demand and supply elasticities. There are, however, two important differences between the methods:

1. The complicated nature of the apparatus involved in Dr. Leontief's method serves strongly to disguise the appearance of fictitious or meaningless cases. Professor Pigou's method, on the other hand, is not dependent on any complicated mathematical or statistical proc-

[10] Most commodities that yield to this method are probably of this kind (e.g., sugar, used in the illustration below). The case considered here is actually that which Dr. Cassels had in mind in the latter part of his article (*op. cit.*, pp. 583–584).

ess; it is easily and rapidly carried out and the fictitious or meaningless cases are more obvious.

2. The method of Dr. Leontief uses the whole material to provide single estimates of the elasticities of demand and supply. It follows that, if a fictitious case is detected, the whole material must be rejected. This is not true of Professor Pigou's method, which consists essentially of a large number of observations of the elasticity of demand or supply from separate triads. Any one triad may display fictitious determinateness or be meaningless and so may be rejected. But, even if a proportion of the triads are rejected in this way, it is probable that *enough observations remain for a final estimate of the demand and supply elasticities.* This conclusion is supported by the example below. It must be remembered that, for a significant result in Professor Pigou's method, it is only necessary that one of the demand and supply shifts should be nearly regular and one fairly irregular.

It appears, therefore, that Professor Pigou's method is to be preferred in almost all cases. Dr. Leontief's method has the advantage only when absolute reliance can be placed in the accuracy of the data used and in the applicability of the assumptions made.[11] And this is not a situation in which economic statisticians are likely to find themselves for some time to come.

The generalized method suggested here can be illustrated in its application to one of the materials used by Professor Pigou. It will suffice to take Professor Schultz' figures giving the yearly consumption and price of sugar in U.S.A. for the period 1890–1914.[12] Since the object here is the same as that of Professor Pigou—to illustrate the method—the figures are allowed to stand without external examination.

The following tentative results appear from the appended table:

I. Three triads (1901, 1907, 1910) show regular shifts in both demand and supply over the three years. The slopes of BB' are regarded as fictitiously determinate.

II (1). Eight triads (1894, 1896, 1897, 1898, 1902, 1905, 1911, and 1912) show regular demand shifts combined with irregular supply shifts. Two other triads (1906 and 1913) are possibly also of the same nature. The slopes of BB' are *estimates of the demand elasticity*, and in ascending order of (negative) magnitude they are: -0.20, -0.22, -0.24, -0.34, -0.50, -0.51, -0.53, -0.54, -0.65, -0.85. The median is -0.505, but without the two doubtful cases the median is -0.52.

[11] See Professor Frisch, *op. cit.*, pp. 30 and 37.

[12] Pigou, *op. cit.*, Table I., p. 396; the figures are obtained from Schultz, *Statistical Laws of Demand and Supply*, p. 213. There is one important numerical error in Professor Pigou's computations of elasticity of demand (the 13th figure).

II (2). Five triads (1892, 1895, 1899, 1908, and 1909) show regular supply shifts combined with irregular demand shifts. Two other triads (1893 and 1900) are possibly also of the same nature. The slopes of BB' are *estimates of the supply elasticity*, and in ascending order of magnitude they are: −0.09, 0, 0.07, 0.27, 0.34, 0.38, 0.46. The median is 0.27, but without the two doubtful cases the median is 0.34.

TABLE 1.—U. S. A. CONSUMPTION (000 SHORT TONS) AND N. Y. WHOLESALE PRICE (CENTS PER LB.) OF SUGAR, 1890–1914

Year	Logarithm of price	Logarithm of consumption	Triads of observations (ABC) centered at each year, 1891–1913			
			Distance $AB'=B'C$	Slope of AC	Distance BB'	Slope of BB'
1890	0.79036	3.21854				
1891	0.66661	3.32552	.0907	−0.649	.0747	−1.210
1892	0.63809	3.31723	.0094	0.223	.0391	0.273
1893	0.68502	3.32960	.0213	−1.539	.0588	−0.094
1894	0.61490	3.35295	.0337	−0.144	.0411	−0.504
1895	0.61826	3.33925	.0215	−0.278	.0189	0.457
1896	0.65629	3.34143	.0219	0.739	.0232	−0.527
1897	0.65350	3.36530	.0203	0.237	.0296	−0.850
1898	0.69592	3.35083	.0192	0.039	.0277	−0.655
1899	0.69188	3.36680	.0269	1.490	.0200	0.337
1900	0.72591	3.39550	.0293	5.053	.0283	−0.003
1901	0.70329	3.42439	.0498	−0.818	.0161	−0.164
1902	0.64885	3.45849	.0242	−0.849	.0404	−0.512
1903	0.66633	3.45576	.0221	1.097	.0193	−7.349
1904	0.67870	3.49122	.0280	0.254	.0322	−1.933
1905	0.72066	3.46953	.0141	−0.625	.0614	−0.541
1906	0.65466	3.50623	.0386	−1.049	.0402	−0.221
1907	0.66736	3.52543	.0307	1.138	.0085	0.513
1908	0.69522	3.55242	.0191	3.430	.0241	0.382
1909	0.67806	3.56217	.0109	16.770	.0178	0.067
1910	0.69653	3.57426	.0257	0.247	.0088	−0.908
1911	0.72795	3.57449	.0102	3.267	.0300	−0.338
1912	0.70252	3.59384	.0539	−0.495	.0234	−0.201
1913	0.63124	3.62242	.0221	−0.956	.0569	−0.241
1914	0.67052	3.62449				

III. Three triads (1891, 1903 and 1904) show very irregular shifts in both demand and supply. The slopes of BB' are entirely without meaning in terms of demand or supply elasticity. The four doubtful cases included in II may also belong to this class.

London School of Economics

CIRCULATION PLANNING: PROPOSAL FOR A NATIONAL ORGANIZATION OF A COMMODITY AND SERVICE EXCHANGE*

By Ragnar Frisch

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* This paper will be supplemented with a mathematical appendix which will appear in the early number of Econometrica.

THE PROBLEM

THE most striking paradox of great depressions, and particularly of the present one, is the fact that poverty is imposed on us in the midst of a world of plenty. Many kinds of goods are actually present in large quantities, and other kinds could without any difficulty be brought forth in abundance, if only the available enormous productive power was let loose. Yet, in spite of this technical and physical abundance, most of us are forced to cut down consumption. We are compelled to make real sacrifices in order to economize in the use of these very goods and services that *could* easily be produced in abundance if we would only use our resources.

A full recognition of the monstrosity of this situation is the first and basic condition for any intelligent discussion of ways and means to get out of the depression. Of course this implies the conclusion that the cause of great depressions, such as the one we are actually in, is in some way or another connected with the present *form of organization* of industry and trade. The depression is not a real poverty crisis, not due to an actual shortage of real values. This must be admitted by everybody, quite regardless of political color.

One very important aspect of the disastrous effects produced during great depressions by the present organization of industry and trade is what might be called the *incapsulating phenomenon*. It manifests itself in various forms and in different economic fields, but its nature and deplorable consequences are always the same. It is fundamentally connected with the fact that modern economic life has been divided into a number of *regions* or *groups*.

Under the present system, the blind "economic laws" will, under certain circumstances, create a situation where these groups are forced *mutually to undermine each other's position*. Each group is forced to curtail the use of the goods produced and services rendered by the other groups, which, in turn, will cause a still further contraction of the demand for its own products, and so on. This meaningless vicious circle is what I understand by the incapsulating phenomenon.

One of the most spectacular examples of this phenomenon is furnished by the protectionist tendencies in international trade relations during a great depression period.

However, it is not only in the international field that such a tendency is at work. *Its effects within the borders of each nation are probably even greater*. It produces a far-reaching stagnation of the national circulation. The bootmaker finds that he cannot afford to buy a new suit because he is unable to sell any shoes, and at the same time the tailor finds that he cannot afford to buy shoes because he is unable to sell any suits. If the effect of this national incapsulating phenomenon could

be calculated in money, it would amount to an enormous sum. It concerns, indeed, all of us, in nearly every phase of our daily life.

The reason for this national circulation stagnation is not, of course, that the parties involved have no desire for the goods and services offered. On the contrary, the desire is in most cases very great. But the reason is that the parties are *unable* to buy or that they do not *dare* to do so. They lack the means or the courage, mainly because *the others* lack the means or the courage.

Unfortunately, the importance of these incapsulating tendencies within the national border is much less clearly recognized by the general public—and even by many professed economists—than the corresponding phenomena in the international field. The reason for this lack of attention to the national aspect of the problem is perhaps that the decisions and reactions by which the innumerably smaller or larger groups in a country yield to the pressure of the forces that tend to contract circulation are less spectacular, and have much less chance of being utilized for sensational front page headlines, than the decisions and happenings in the field of international trade relations. And, as is well known, the attention of the general public is not proportional to the real importance of the phenomena but to the space devoted to them in the newspapers.

Within the national border, the possibilities of doing something constructive to combat the devastating effects of the circulation stagnation are much greater than internationally. Experience has amply proved that very little can at present be obtained by international agreement. Of course this in itself puts certain limits to the possible solution of the circulation problem. A *complete* solution would necessitate international co-operation. But this fact must not make us overlook that great improvements can be achieved by eliminating at least that part of the circulation stagnation which is national. At the moment it seems more realistic to work for this partial solution. There is all the more reason for this attitude as the elimination of the national circulation stagnation will not work counter to a final international solution, but rather prepare the way for it. In the present paper I shall be concerned primarily with the national aspect of the problem.

In the attempt at introducing a certain measure of planning in exchange activity, one is confronted with an optimum problem of great complexity. The prime object of the planning must be to utilize more fully the existing productive capacity, but at the same time the activity must be guided in such a way as to conform as much as possible with the particular desires of the individuals and groups involved. These two objects are in many cases mutually exclusive. The weighing

of their relative importance is precisely the fundamental problem of the planning.

Certain aspects of this optimum problem are of a theoretical nature. We are indeed here confronted with problems very similar to those we meet in productivity theory, only the corresponding problems in circulation planning are more far-reaching and complicated. Furthermore, most of these problems are quantitative: *how much* should this activity be increased and that other curtailed? All of which means that these problems are essentially *econometric*.

The mathematical aspect of these problems, and the nature of the computation technique involved, is in certain respects similar to that used in actuarial science, but much broader in scope, because the actuaries of national planning must utilize, to a considerable extent, also that type of analysis that is developed in modern economic theory.

Parts I and III of the present paper attempt to make some contribution towards the analysis of these mixed problems where technical-computational and economic-theoretical considerations must concur in determining the final solution. In Part II, I shall make some remarks on the practical aspect of the circulation scheme proposed.

PART I. THEORY

1. Intrinsic Tendencies towards Contraction and Expansion in an Exchange System

I shall first give an analysis of a simplified theoretical system showing the intrinsic tendency to contraction or expansion that may exist in an economy based on *exchange* of commodities and services. This analysis will exhibit the nature of the maladjustment which it is the object of the circulation planning to overcome.

Suppose for simplicity that there are only two persons (or groups), say a shoemaker and a farmer. Let us imagine that they exchange their products at regular intervals, say at fairs held one or twice a year, as was commonly the case in the old days. This assumption is only made in order to simplify the exposition; the essential features of the development will be the same if the exchange activity takes place in a more continuous fashion. For simplicity we shall further assume that the prices are constant. The quantities sold or bought can therefore simply be measures in dollars' worth.

It seems reasonable to assume that the shoemaker will at any given moment buy all the more of the product of the other person, the more he has himself been able to sell at the moment immediately preceding.

And similarly for the farmer. To construct a simple example, we may, for instance, assume that the amount bought by the shoemaker is *proportional* to the amount he sold at the previous moment, and similarly for the farmer. That is, we put

$$\begin{aligned} a_t &= \alpha b_{t-1} \\ b_t &= \beta a_{t-1} \end{aligned} \tag{1.1}$$

where a_t and b_t are the amounts bought by the shoemaker and the farmer respectively; and α and β are constants; α and β are a kind of "coefficient of optimism" expressing how the previous sales influence each man's further policy.

If the exchange starts at the point of time $t=0$ where a (the shoemaker) buys a_0 and b (the farmer) buys b_0, the further development of the exchange activity will be as follows:

$$\begin{aligned} a_1 &= \alpha b_0 \\ b_1 &= \beta a_0 \\ \hline a_2 &= \alpha\beta a_0 \\ b_2 &= \alpha\beta b_0 \\ \hline a_3 &= \alpha\cdot\alpha\beta b_0 \\ b_3 &= \beta\cdot\alpha\beta a_0 \\ \hline &\text{etc.} \end{aligned}$$

Quite generally, we get

$$\begin{array}{cc} a_t = a_0\gamma^t & a_t = \alpha b_0\gamma^{t-1} \\ b_t = b_0\gamma^t & b_t = \beta a_0\gamma^{t-1} \\ \text{(when } t \text{ is even)} & \text{(when } t \text{ is odd)} \end{array} \tag{1.2}$$

where

$$\gamma = \sqrt{\alpha\beta} \tag{1.3}$$

In a single formula this may be written

$$\begin{aligned} a_t &= (A_1 + (-)^t A_2)\gamma^t \\ b_t &= (B_1 + (-)^t B_2)\gamma^t \end{aligned} \tag{1.4}$$

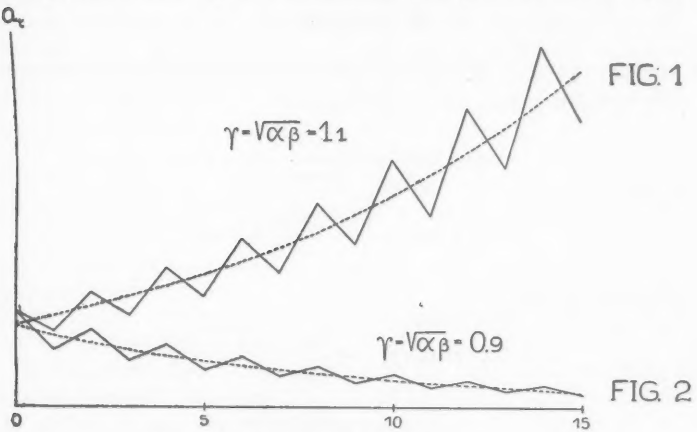
where

$$\begin{aligned} A_1 &= 1 + \frac{1}{2}\left(\sqrt{\frac{\alpha}{\beta}}\cdot\frac{b_0}{a_0} - 1\right) & A_2 &= -\frac{1}{2}\left(\sqrt{\frac{\alpha}{\beta}}\cdot\frac{b_0}{a_0} - 1\right) \\ B_1 &= 1 + \frac{1}{2}\left(\sqrt{\frac{\beta}{\alpha}}\cdot\frac{a_0}{b_0} - 1\right) & B_2 &= -\frac{1}{2}\left(\sqrt{\frac{\beta}{\alpha}}\cdot\frac{a_0}{b_0} - 1\right). \end{aligned} \tag{1.5}$$

The formula (1.4) shows that the development of, say, a_t is described by a time series that has two components: The main component is simply the exponential trend $A_1\gamma^t$, and the second component is a cycle with a period of two time units, in other words, it alternates between a maximum at one moment and a minimum at the other.

The exponential trend that forms the main component will be increasing or decreasing accordingly as $\gamma > 1$ or $\gamma < 1$. But γ as defined by (1.3) can simply be looked upon as an *average* coefficient of optimism. Thus we may say that if the two parties are on the average in the "spending mood," meaning by this that $\gamma > 1$, then the system will expand, but if they are on the average in the "saving mood," i.e., if $\gamma < 1$ *the whole system will gradually dwindle down to nothing.*

The amplitude of the *short* cycle is always proportional to the trend ordinate. This short cycle is simply a reflex of the fact that the exchange started in a way that was not in harmony with the tendency to



FIGURES 1 and 2

further development represented by the coefficients of optimism. More precisely, this maladjustment in the initial conditions can be expressed by saying that

$$\frac{a_0}{\sqrt{\alpha}} \neq \frac{b_0}{\sqrt{\beta}}. \tag{1.6}$$

This is obviously the condition that the terms A_2 and B_2 shall be present in (1.3). The short cycle here considered is only an incidental, secondary feature of the development.

Figures 1 and 2 indicate the time shape of the solutions that will

follow from two alternative sets of values of the structural constants α and β and of the initial conditions a_0 and b_0.

In one respect the above scheme is too simple: If the parties are in the deflationary mood, expressed by the fact that the average coefficient of optimism γ is less than unity, it is not likely that the ensuing downward movement (as exhibited in Figure 1) will continue straight down to zero. Sooner or later another tendency will come into play: The parties will have a certain minimum of existence and will therefore be forced to *modify* their policy of curtailing purchases in strict proportion to the decrease in their own sales. They will reach a point where they simply *cannot* restrict purchases any more, no matter what the further consequences may be. To express this, we may, for instance, describe their policy by equations of the form

$$\begin{aligned} a_t &= a_* + \alpha b_{t-1} \\ b_t &= b_* + \beta a_{t-1} \end{aligned} \tag{1.7}$$

where a_* and b_* are constants depending on the minimum of existence. This leads to

$$\begin{aligned} a_t &= a_0\gamma^t + \frac{a_* + \alpha b_*}{1-\gamma^2}(1-\gamma^t) \\ b_t &= b_0\gamma^t + \frac{b_* + \beta a_*}{1-\gamma^2}(1-\gamma^t) \end{aligned} \tag{1.8}$$

when t is even, and

$$\begin{aligned} a_t &= (a_* + \alpha b_0)\gamma^{t-1} + \frac{a_* + \alpha b_*}{1-\gamma^2}(1-\gamma^{t-1}) \\ b_t &= (b_* + \beta a_0)\gamma^{t-1} + \frac{b_* + \beta a_*}{1-\gamma^2}(1-\gamma^{t-1}) \end{aligned} \tag{1.9}$$

when t is odd.

In this case the contraction of the exchange, which takes place if $\gamma < 1$, will not completely eliminate all activity, but will bring it down to the stationary levels.

$$a_\infty = \frac{a_* + \alpha b_*}{1-\gamma^2} \tag{1.10}$$

$$b_\infty = \frac{b_* + \beta a_*}{1-\gamma^2}. \tag{1.11}$$

2. The Indebtedness Factor in Exchange Contraction and Expansion

So far nothing has been said about the monetary and credit relations between the two parties. The preceding analysis assumes of course that some form of credit is used, otherwise we would always have $a_t = b_t$ so that the theoretical set-up would be reduced to a perfectly static one. The simplest assumption about the credit transaction is that a owes to b, or vice versa, the total difference between the purchases and sales up to the instance considered. It is easy to see that, in the contraction case illustrated in Figure 1, this total indebtedness will converge towards a definite magnitude, while in the expansion case it will increase towards infinity.

The latter consequence shows that the theoretical scheme of Section 1 is too simple to describe the long-time effects that take place in the system. In order to make the scheme more realistic, we ought to allow the magnitude of the existing debt to exert an influence on the buying policy of the parties at a given instant. Professor Irving Fisher has recently drawn attention to the rôle played by debts in the dynamics of the business cycle.[1] He emphasizes in particular the fact that attempts at paying off debts during a depression period will generally *increase* the real debt burden, because the paying off of nominal debts will, through its deflationary effect, increase the value of the monetary unit more rapidly than the nominal debts are brought down. Undoubtedly this is a very important aspect of the problem. In the present connection we shall, however, not follow up particularly the effect of changing prices, but rather look into the effect on the buying activity that will in general be produced by the indebtedness factor quite irrespective of any price changes. These effects are of particular importance for our problem. They will be *magnified* by the fact pointed out by Professor Fisher.

Let us again study the evolution that will take place in a certain simplified system. Let us assume that the buying policies of the parties are defined by equations of the form

$$\begin{aligned} a_t &= \alpha b_{t-1} - \lambda G_{t-1} \\ b_t &= \beta a_{t-1} + \mu G_{t-1} \end{aligned} \tag{2.1}$$

where α, β, λ and μ are non-negative constants, and

$$G_t = (a_0 + a_1 + \cdots + a_t) - (b_0 + b_1 + \cdots + b_t) \tag{2.2}$$

is the total sum which a owes to b. In this case an uninterrupted long-time increase in the volume of transactions is not likely to take place.

[1] See his book *Booms and Depressions*, New York, 1933, and his paper in ECONOMETRICA, October 1933.

Indeed, any difference between the two parties' ways of acting that results in an increasing indebtedness from one to the other will increase the second terms in (2.1) and this will probably after a while put a brake on further expansion. A full understanding of the forces at work in this case, and of the time shape of the resulting curves, cannot, however, be obtained by a verbal discussion; we have to analyse the situation with the appropriate mathematical tools.

We first notice that it is possible to deduce from the equations (2.1) a certain higher order difference-equation in the single variable a_t. Writing down the equations (2.1) both for t and $t-1$, and subtracting, we get

$$a_t - a_{t-1} = \alpha(b_{t-1} - b_{t-2}) - \lambda(a_{t-1} - b_{t-1})$$
$$b_t - b_{t-1} = \beta(a_{t-1} - a_{t-2}) + \mu(a_{t-1} - b_{t-1}).$$

Replacing t by $t+1$ in the first of these equations and bringing the terms with b over to the left side, we get

$$(\alpha + \lambda)b_t - \alpha b_{t-1} = a_{t+1} - (1 - \lambda)a_t$$
$$b_t - (1 - \mu)b_{t-1} = (\beta + \mu)a_{t-1} - \beta a_{t-2}.$$

If the determinant

$$\begin{vmatrix} \alpha + \lambda & \alpha \\ 1 & 1 - \mu \end{vmatrix}$$

is $\neq 0$, this system may be solved with respect to b_t and b_{t-1}. Both in the expression thus obtained for b_t and in that for b_{t-1} there will occur certain a's. Consequently, if in the subscripts on the a's in the latter expression we replace $t-1$ by t, we ought to obtain the former expression. This entails a certain consequence on the a's. The condition turns out to be that a_t must satisfy the fourth order difference equation

$$(2.3)\quad a_{t+4} - (2 - \lambda - \mu)a_{t+3} - [(\alpha + \lambda)(\beta + \mu) - (1 - \lambda)(1 - \mu)]a_{t+2} + [(\alpha + \lambda)\beta + \alpha(\beta + \mu)]a_{t+1} - \alpha\beta a_t = 0.$$

Since the coefficients of this equation do not change if the symbols α and β, λ and μ are interchanged, we conclude that b_t must satisfy the same equation.

As is well known from the theory of difference equations, the time shape of the solution a_t of an equation of the form (2.3) depends on the roots of the corresponding characteristic equation, i.e., the algebraic equation obtained by replacing a_{t+4}, a_{t+3}, etc., by z^4, $z^3 \cdots$ etc., z being the unknown to be determined from the equation. It is easily seen that $z=1$ is a root of the equation thus obtained. Dividing out by $(z-1)$, we thus find that the characteristic equation may be written

$$(2.4)\quad z^3 + c_1 z^2 + c_2 z + c_3 = 0$$

where

$$c_1 = -(1 - \lambda - \mu)$$

$$c_2 = -(\lambda\beta + \alpha\mu + \alpha\beta) \tag{2.5}$$

$$c_3 = \alpha\beta.$$

The root $z=1$ means that a_t will contain as one of its components a *constant* $\bar{a}$. The nature of the other components is determined by the roots of (2.4).

Before we proceed to a discussion of these in the general case, let us apply the equation to the special case where the buying policy is given by (1.1). In this case we have $\lambda=\mu=0$, which by (2.4) gives the characteristic equation.

$$z^3 - z^2 - \alpha\beta z + \alpha\beta = 0$$

that is

$$(z - 1)(z^2 - \alpha\beta) = 0. \tag{2.6}$$

Thus in this case one more root becomes equal to 1, and the two other roots are $z=+\gamma$ and $z=-\gamma$, γ denoting as before the constant $\sqrt{\alpha\beta}$. This shows that, apart from a linear term in t (which here vanishes since by the initial conditions both its constants are determined to be zero), the time function a_t will contain the two components γ^t and $(-\gamma)^t$ each multiplied by a constant to be determined by the initial conditions. This leads to the solution (1.4) which we originally found by a direct elementary method.

Now as to the general equation (2.4). Its discriminant is

$$D = c_1^2c_2^2 + 18c_1c_2c_3 - 4c_2^3 - 4c_1^3c_3 - 27c_3^2. \tag{2.7}$$

The time shape of the solution will depend primarily on the sign of this discriminant; if it is positive, the characteristic equation will have three real roots and the system consequently evolve exponentially. But if the discriminant is negative, cycles will occur. The accompaning table indicates the value of D for $\alpha=0.9$, $\beta=0.8$, $\mu=0.1$ and different values of λ

(2.8)

	Discriminant D
$\lambda = 0.2$	-0.097
0.3	-0.182
0.4	-0.224
0.6	-0.167
0.7	-0.062
0.8	$+0.127$
0.9	$+0.330.$

This shows that both cyclical and exponential solutions are possible within the range of values which it is plausible to attribute to the structural parameter α, β, λ and μ.

If the discriminant is negative, that is, if the equation (2.4) has one real and two conjugate imaginary roots, the real root must be negative. This follows from the fact that in our case, we must always have $c_3 > 0$ since both α and β must be assumed positive.

Let $z = -\kappa$ be the negative root, and let $z = e^{-\theta}(\cos\omega \pm i \sin\omega)$ be the two conjugate imaginary roots. In other words, $-\theta$ is the natural logarithm of the modulus of the complex roots, and ω their phase angles in the Gaussian plane. From the classical theory of linear difference equations, it follows that the time shape of a_t and b_t will then be of the form.

$$(2.9) \qquad a_t = \bar{a} + e^{-\theta t}(A_1 \sin \omega t + A_2 \cos \omega t) + (-)^t A_3 \kappa^t$$

$$(2.10) \qquad b_t = \bar{b} + e^{-\theta t}(B_1 \sin \omega t + B_2 \cos \omega t) + (-)^t B_3 \kappa^t$$

where $\bar{a}$ and $\bar{b}$ as well as the A's and B's are constants to be determined by the initial conditions.

The last terms in (2.9) and (2.10) represent a short cycle which alternates between a maximum at one moment and a minimum at the next. This oscillation is of the same nature as the short cycle we discussed in the simple case (1.1). It is not essential in the present connection.

The first terms in (2.9) and (2.10) represent the constant levels around which the fluctuations take place; these terms need not occupy us very much either.

The interesting feature of the movement which the present theoretical set-up will give rise to is the cyclical movement represented by the middle terms of (2.9) and (2.10).

In all the cases listed in (2.8) the *damping* of this cycle is very heavy, it is indeed so heavy that if the curve is drawn on a usual scale no marked cycles will appear. Is this a general property necessarily connected with such values of the structural parameters as are plausible in view of the concrete meaning of the parameter? To investigate this question we may first notice that it would seem quite unreasonable to assume that the behavior of the parties is such that any of the structural parameters are negative. Consider then for instance the condition $\mu \geqq 0$; we find that this condition entails[2]

$$(2.11) \qquad \cos\omega \geqq \frac{\alpha + \kappa}{\alpha + e^{-2}} \cdot \frac{1 + \kappa}{2\kappa} e^{-\theta}.$$

[2] These limits as well as the numerical solutions of the characteristic equation were worked out by Mr. Harald Holme, assistant in my laboratory in the University Institute of Economics, Oslo.

Putting as a limiting case $\mu=0$, and utilizing the further condition $\beta \geqq 0$ we get

$$\kappa(2\cos\omega - e^{-\theta}) \geqq e^{-\theta}. \tag{2.12}$$

Assuming here κ to be unity (which means that the secondary feature of the development, namely the *short* cycle, shall neither dampen away nor "explode," but just go on with the same intensity) the inequality (2.12) reduces to

$$\cos\omega \geqq e^{-\theta}. \tag{2.13}$$

By means of these inequalities it is easy to find examples of sets of values of the structural parameters leading to cycles *that are not heavily damped.* In Table II are given some such examples.

TABLE I

Example No.	If the structural parameters are				The time shape characteristics will be		
	α	β	λ	μ	Length of cycle $p=2\pi/\omega$	Damping factor pr. cycle $e^{-\theta p}$	Real root $-\kappa$
1	2.00	0.50	0.56	0.04	20	0.10	−1.3
2	1.50	0.584	0.274	0.0262	30	0.05	−1.07
3	2.00	0.376	0.28	0.013	22	0.05	−0.98
4	2.00	0.36	0.25	0.05	30	0.01	−0.95
5	3.00	0.29	0.29	0.003	30	0.05	−0.95
6	2.048	0.433	0.158	0	30	0.165	−1.0
7	6.428	0.1465	0.102	0	30	0.407	−1.0
8	2.734	0.364	0.0088	0	120	0.698	−1.0
9	1.350	0.739	0.0023	0	360	0.698	−1.0

Consider example No. 5. Here a (the shoemaker) is in a very expansive mood ($\alpha=3.0$), while b (the farmer) is very much for saving ($\beta=0.29$); at the same time a is rather sensitive to changes in his debt situation ($\lambda=0.29$), while b is not ($\mu=0.003$). The result is a cycle that has a good amount of damping, the damping is, however, far less extreme than in the examples of (2.8). After the lapse of a length of time equal to one cycle (30 "years") there is still 5 per cent of the amplitude left.

In the next example, No. 6, there is less difference between the "expansiveness" of the two parties, and they are both less sensitive to changes in their debt situation. This turns out to produce a situation where the damping is less heavy; after one cycle there is now left 16.5 per cent of the amplitude. In example No. 9, this tendency is still more pronounced. In this example the difference in the expansive-

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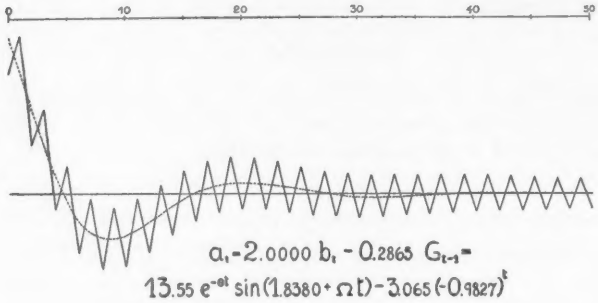
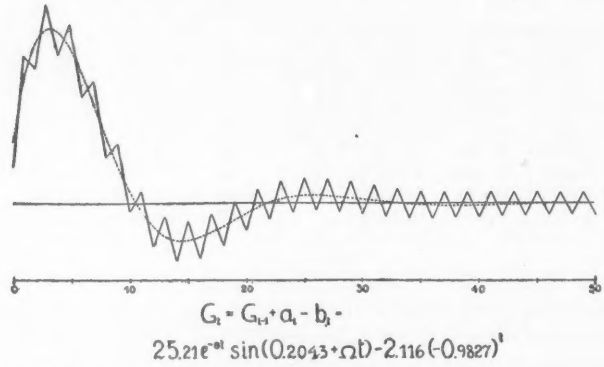
ness of the parties is not great ($\alpha = 1.35$, $\beta = 0.739$) and none of them is very sensitive to changes in the debt situation. After all, the last example seems to be the most plausible. *And in this case there appears a cycle that is very lightly damped,* in the course of each cycle the curve only loses about 30 per cent of its amplitude, so that the swings will continue for a considerable span of time. The development of the system in this case is indicated in Figure 3.





Figures 1 and 2 represent examples Nos. 2 and 3 respectively. A comparison between these two figures exhibits the rôle played by the real root: In Figure 1, where κ is larger than unity, the short fluctuations are increasing, while in Figure 2, where κ is less than unity, they dampen out.

In examples Nos. 8 and 9 the periods appear much longer than in the other examples. Since the unit of measurement for time in the examples is arbitrary, the long periods in the examples in question can simply be looked upon as expressing the fact that the exchange acts

here take place *more frequently*. Also in this respect are these latter examples more realistic than the others.

All the cycles here obtained dampen out if we consider a sufficiently long span of time, but as I have pointed out elsewhere,[3] when a system obeying such dynamic laws is hit by erratic impulses of various sorts, for instance accidental disturbances which cause the parties in-

$$a_t = 1.3505\, b_{t-1} - 0.0023\, G_{t-1} - 9.7308\, e^{-\alpha t} \sin(1.5891 + \Omega t) + 0.27086\,(-1)^t$$

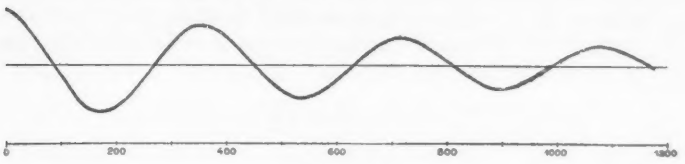


FIGURE 3

volved to *deviate* every so often a little from the underlying dynamic laws of the system, then a *maintained* oscillation will be created whose period is somewhat changing but on the average equal to the period of the original undisturbed solution. Figure 4 indicates the result obtained if the system in example No. 9, is maintained by a stream of erratic shocks. The "shocks" by means of which Figure 4 was constructed were taken from end-digits in the Norwegian State lottery.

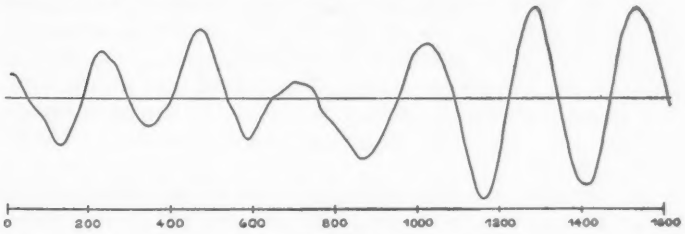


FIGURE 4

This analysis leading up to the result exhibited in Figure 4, shows that violent depressions may be caused by the mere fact that the parties involved have a certain typical *behavior* in regard to buying and loaning. These cycles are pure circulation-cycles. They have nothing

[3] "Propagation Problems and Impulse Problems in Dynamic Economics," in the volume published in honor of Gustav Cassel, London, 1933. A more extensive analysis is given in my forthcoming book *Changing Harmonics* to be published as a monograph of the Cowles Commission for Research in Economics.

to do with the various technical features of the production process, or with real investment activity. Indeed, no such considerations have been introduced in the preceding analysis.

Of course I do not maintain that the technical features do not in reality play some rôle in the course of business cycles (my paper in the Cassel volume was devoted to cycles produced by certain peculiarities of capitalistic production); but I do maintain that there exist also certain other important phenomena that are connected solely with the circulation and exchange activity and act more or less independently to *aggravate* the complex fluctuations which we know from experience. It is a regulation of these circulation factors that is the aim of the plan developed in the subsequent Sections.

3. A Simplified Example of a Planned Exchange

Consider three persons (or groups): a shoemaker, a tailor and a farmer. Suppose that they are in a heavy depression period of the sort described in the preceding Section. This means these three persons are mutually waiting for each other's orders, none of them daring to make a purchase, because they do not feel assured that they will be able to sell their own products.

Now suppose that an organizer goes to the shoemaker and puts the following question up to him: How much will you buy from the tailor and the farmer respectively if you are *assured* an increase in the sales of your own products equal to the total sum of what you buy from the others. The shoemaker answers, for instance, that if he is assured an increase in his own sales of \$150 he will buy \$50 worth from the tailor and \$100 worth from the farmer, making a total of \$150. The condition for the shoemaker's choice is thus that the total sum by which his sales are guaranteed to increase will be used by him for purchases; in other words, he will not have an opportunity to *save* any part of the increase in sales.

Our organizer will put similar questions to the tailor and the farmer. Suppose that the answers are as indicated in (3.1). Such a table may perhaps be called a *request-table* or a *request-matrix*. In (3.1) it is

(3.1)	Wants to buy from			Total
	Shoemaker	Tailor	Farmer	
Shoemaker	0	50	100	150
Tailor	40	0	180	220
Farmer	110	170	0	280
Total	150	220	280	650

assumed that the demand for shoes is equal to the supply, and similarly for the tailor's products and for farm products. Of course, in reality, this condition will hardly ever be exactly fulfilled; one of the big problems of circulation planning is just *how* such a balance shall be brought about: the greater part of this paper is devoted to this question. For a moment let us assume, however, that the demand-supply balance exists.

If this is so, it is, of course, an easy matter for our organizer to *force* the circulation to move on. He only has to print \$150 of warrants ("commodity notes") on shoes, of which he gives \$40 to the tailor and \$110 to the farmer. Further, he prints \$220 warrants on clothing, of which he gives the shoemaker \$50 and the farmer \$170. Finally, he prints \$280 warrants on farm products, of which he gives the shoemaker \$100 and the tailor \$180. Each of the parties has thus got warrants on just the goods, and for exactly the amounts, they asked for.

If the organizer announces that these warrants are valid for a certain short period, say a month, it is clear that all three parties concerned will be almost certain to use the warrants. The probability that some of the warrants will not be used will hardly be larger than the probability that a lottery-stake will remain uncalled for.

Further, it is obvious that, during the process, each of the three parties will see his sales increase by just the amount intended, so that, when the process is completed, each person will have acquired from his customers a sum in warrants just equal to the sum he received at the beginning from the organizer. When the process is completed the organizer may, therefore, again visit all the three parties and collect from each an amount in warrants equal to the amount which this person originally received. The organizer will bring all the warrants home and *burn them*. The cycle is closed, the "money," or the warrants, have done their service and may be destroyed. But the economic *reality* is obtained: the various parties have used their productive faculty and have thus satisfied their own and the other parties' wants.

This indicates in all its simplicity the underlying idea of a system of circulation planning. When this idea is to be worked out, various special problems arise, the most important of which will be discussed in the following Sections.

4. The Correction of the Request Matrix by the Principle of Partakers' Percentages

The circulation scheme the underlying idea of which is indicated in Section 3 is built on a *voluntary* adherence from the various persons or groups. This being the case, the request-matrix, as determined by questioning the parties, will never, of course, show an exact demand-

supply balance, that is, it will not have row-column equality. The sum in the first row will not be exactly equal to the sum in the first column, the sum in the second row not equal to that in the second column, etc. The problem thus arises: In what way can modifications be introduced that will bring about such a balance? From the purely formal point of view this may be done in an infinity of ways, indeed, there are n^2 elements that may be corrected, and n conditions imposed, of which only $n-1$ are independent. In this general setting the solutions has therefore, a great number of degrees of freedom.

But this purely formal aspect of the problem has little interest from the economic viewpoint. The crux of the matter, so far as applications are concerned, is that the correcting of the request-matrix must be done in such a way as to leave the originally requested quantities *as far as possible unchanged.* This should be done in order to satisfy the expressed desires of the parties to the largest possible extent. This ought to be the fundamental optimum consideration in the planning of the circulation.

Which features of the matrix should be changed, and to what extent the corrections should be carried through, will depend on a *weighing* of the various desires towards each other. Of course, it is hardly possible to find any absolutely *objective* and *exact* scale for making such comparisons. A considerable amount of estimate will always be involved, the scale of comparison must be constructed more or less by plausibility considerations. But nevertheless some sort of plausible weighing will always be possible. We have here one of those cases—so frequent in economic practice—where it can be "proved" by abstract reasoning that a solution is not possible, but where life itself compels us nevertheless to find a way out. I have been told that during the war a slogan was created in the U. S. Departments: "*It can't be done—but here it is.*" There is much to be learned from this slogan, particularly for the theoretical economist who is facing the problem of constructing a scale of comparison for the desires of various groups.

The correcting of the request-matrix ought to be done by means of certain well defined *principles* that can be applied in a rather mechanical way to any given request-matrix. Whatever estimate and plausibility considerations are necessary should be incorporated in the definition of the principles themselves. From the practical viewpoint such an arrangement seems imperative. If some committee or board of directors should decide each case upon its own merits, confusion would undoubtedly arise. There would be arbitrariness and lack of co-ordination in the handling of the many figures in the request-matrix.

In this and the following sections I shall discuss certain principles

for the correction of the request-matrix that may be applied either separately or in conjunction.

Many—if not most—of the persons or groups who are likely to take part in the planned exchange will probably consider it a fair and straight arrangement if the necessary adjustment is made on a strict percentage basis for each person, so that the *relative* distribution of the purchases for any given person is maintained unchanged while his total volume of purchase is adjusted. This means that there is prescribed for each person a certain *percentage* by which he has to reduce all his figures. This we shall call *the principle of partakers' percentages.*

This principle seems not only fair and straight, but, for many partakers, it will be the only natural solution. We only have to think of an enterprise that requests certain factors of production. To the extent that the coefficients of production are constant for variations in output of the order of magnitudes here considered, the principle of partakers' percentages would be the only correct solution.

Does there exist any set of partakers' percentages that will, when applied to the original request-matrix, furnish a matrix with complete row-column equality? Such a set of percentages always exists, and it is in general uniquely determined, apart from an arbitrary factor of proportionality that is common to all the percentages. This is easily proved by the lemmas of Section 21.

Indeed, let z_i be the reduction percentage applied for partaker No. 1. The elements of the corrected matrix will then be

$$(1 + z_i)a_{ij} \tag{4.1}$$

where a_{ij} are the elements of the original request-matrix.

If such a set of percentages z_i is given, the total purchase c_i of the partaker No. 1 is also given; it is simply equal to

$$c_i = (1 + z_i)a_{i0} \tag{4.2}$$

$a_{i0} = \sum_k a_{ik}$ being the total request made from partaker No. 1. The magnitudes c_i will be called the *volume* numbers. Vice versa, if the quantities c_i are given, the percentages z_i are also determined. The problem may therefore just as well be formulated in terms of the c's as in terms of the z's; this transformation is convenient and will be used in the following. In terms of the c's the elements of the corrected matrix are

$$c_i\alpha_{ij} \tag{4.3}$$

where

$$\alpha_{ij} = \frac{a_{ij}}{a_{i0}} \tag{4.4}$$

are the *percentages* with which partaker No. 1 originally proposed to distribute his purchase over the various groups. Of course we have

$$\alpha_{i0} = \sum_k \alpha_{ik} = 1. \tag{4.5}$$

The sum of the i-th column in the corrected matrix is $\sum_k c_k \alpha_{ki}$ and the condition that the corrected matrix shall have complete row-column equality is, therefore, expressed by

$$\sum_k c_k \alpha_{ki} = c_i \tag{4.6}$$

that is,

$$\sum_k c_k(\alpha_{ki} - e_{ki}) = 0 \qquad (i = 1, 2 \cdots) \tag{4.7}$$

where e_{ki} are the unit numbers

$$e_{ki} = \begin{cases} 1 \text{ if } k = i \\ 0 \text{ if } k \neq i \end{cases} \tag{4.8}$$

(4.7) form a system of n homogeneous equations in the n quantities c_k. The matrix of the coefficients has the property that the sum in any row is zero. By (4.5) we get indeed $\sum_i(\alpha_{ki}-e_{ki})=0$. The determinant of the system is consequently zero, so that there certainly exists a solution in the c's. The determinant will not (except by coincidence) be of lower rank; we may therefore conclude that in general there exists a solution in the c's that is uniquely determined apart from an arbitrary factor of proportionality. From the theory of linear equations we know that the c's are proportional to the elements in any row of the adjoint of the matrix $(\alpha_{ki}-e_{ki})$. Since $(\alpha_{ki}-e_{ki})$ is a singular matrix it does not matter which row we pick out, since all the rows in the adjoint of a singular matrix are proportional. Incidentally, it follows from the concluding remark under Lemma II in Section 21 that the rows of the adjoint of $(\alpha_{ki}-e_{ki})$ are not only proportional but *equal*.

Let

$$P_1, P_2 \cdots P_n \tag{4.9}$$

be the numbers obtained from the elements in any row of the adjoint of $(\alpha_{ki}-e_{ki})$ by reducing these elements with a common factor so that their sum becomes equal to unity. Such a reduction is possible since by Lemma III of Section 21 the sum in question is not zero. For the numbers P_i thus defined we have

$$\sum_k P_k = 1. \tag{4.10}$$

The numbers c_i that form the solution of (4.7) are then proportional to the P_i, and by virtue of (4.10) the proportionality factor is nothing but the *sum* of all the c_i. Denoting this sum by

$$C = \sum_k c_k \tag{4.11}$$

we consequently have

$$c_i = CP_i \tag{4.12}$$

Thus there can always be determined a set of numbers c_i that will satisfy (4.7) and therefore produce complete demand-supply equality in the exchange matrix. But that is not yet the answer to our problem. From the practical point of view the essential question is of course if these numbers will be *non-negative*. Only in this case will they have a concrete meaning as *volume numbers* for the partakers and thus furnish a real solution of the problem before us.

The non-negativity of all the magnitudes c_i determined by (4.12) follows from Lemma III in Section 21. Indeed, the matrix $(\alpha_{ki}-e_{ki})$ is equal to the matrix (21.4) apart from the factors $a_{10}, a_{20} \cdots a_{n0}$. All the elements of the adjoint of $(\alpha_{ki}-e_{ki})$ thus have the same sign, the reduced numbers P_i are consequently all non-negative, and hence all the c_i non-negative, provided of course that the arbitrary parameter C (which is equal to the sum of the c's) is selected non-negative. The numbers P_i may be looked upon as positive percentages expressing how the total volume of operation C is distributed over the various partakers; the P_i may therefore be called the *relative volume* members.

The total volume C which so far appears as an arbitrary parameter must be determined by certain supplementary conditions. During the tentative organization period of the exchange system, C may, for instance, be determined as a rather small number, thus making it possible to experiment on a small scale, until it is found safe to let the operations assume larger proportions. Later, when experience is gained, the main concern in determining C will be to assure the fullest utilization of the available capacity, without, however, impelling any of the parties to participate with a larger volume than they originally requested. In certain circumstances this latter condition may impose rather severe restrictions on the total volume of operations. We shall later discuss certain modifications that will in part overcome this difficulty. At present let us study the nature of the limitation that follows from a strict application of the condition considered.

If we require that the *relative* distribution of each person's purchases shall be exactly maintained, and also require that for none of the persons will the total volume of operations after correction be larger than

the volume originally requested by him, then it is clear that all the adaptation must be an adaptation *downwards*. In this process some of the commodities (or services) may act as a troublesome *minimum factor*. Such a situation arises if the supply of a given commodity is small, which means that one of the parties only requests to partake for a small total sum, and if the other parties manifest an effective demand for this commodity. It is clear that if the supply of this commodity is not to be increased, a strict maintenance of the same *relative* distribution as originally claimed by the various parties, is only compatible with a heavy curtailment of the total volume of transactions for the community.

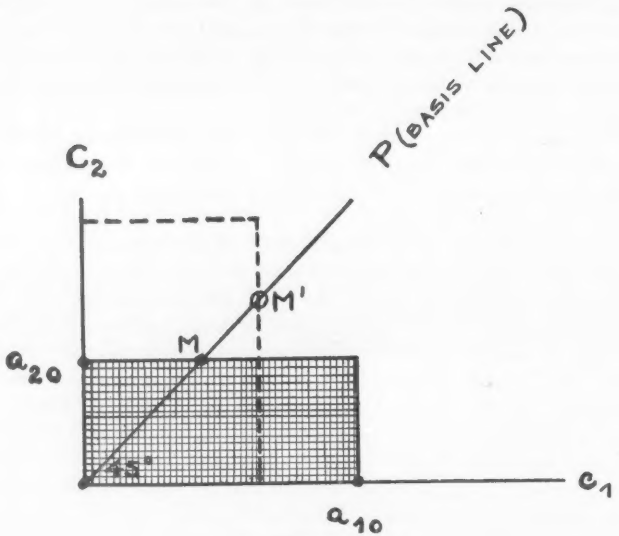


FIGURE 1

The exact point to which the transactions must be brought down may be illustrated graphically as follows. Consider first the case of only two persons (or groups). Let c_1 and c_2 be their total volume of operation after correction of the request matrix. Let c_1 and c_2 be measured along the two axes in a rectangular system of coordinates. (See Figure 1.) Of the various points in this diagram only those lying on the straight line P under 45° are possible. Indeed if the two persons can trade only with each other, the amount which one buys (and sells) must be equal to the amount which the other buys (and sells). The line P we shall call the *basis line*. Trying to make the total volume of operation for

the community as large as possible will mean that we attempt to move on the basis line outwards as far as possible. How far can we go? This is limited by the exchange requests originally made by the person. If No. 1 originally requested a total of a_{10} and No. 2 a total of a_{20}, and c_1 and c_2 shall not surpass these limits, the point (c_1, c_2) is bound to lie somewhere within the shaded rectangle in Figure 1, or on its borders. This means that the point (c_1, c_2) to be finally selected is the point M where the basis line leaves the shaded rectangle. If the point of exit is on the upper (horizontal) side of the rectangle, the request of No. 2 is used to full capacity but not that of No. 1. If the rectangle had been the dotted one, where the point of exit is M'. we would have had the opposite situation.

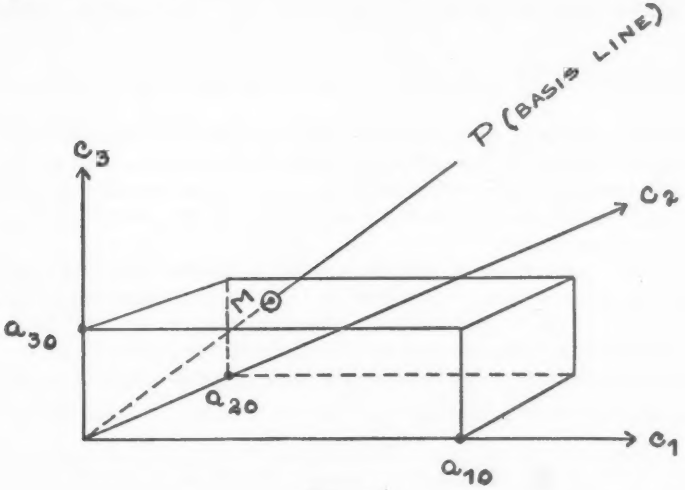


FIGURE 2

By virtue of (4.12) we would expect the direction numbers of the line P to be the numbers P_1 and P_2 determined by the elements of the adjoint of $(\alpha_{ki}-e_{ki})$. In the present case this matrix is

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \text{ and its adjoint consequently } \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$$

so that $P_1=P_2=\frac{1}{2}$ which is in accordance with Figure 1.

A similar representation can be used in three dimensions. See Figure 2.

In the case represented in Figure 2 it is only the request of No. 3 that will be fully utilized.

In the general case of n persons the point of exit M may be determined algebraically as follows. Along the basis line we have $c_i = CP_i$ where C is an arbitrary positive parameter. The boundary condition is that for each i we shall have

$$c_i \leqq a_{i0}$$

hence

$$C \leqq \frac{a_{i0}}{P_i} \text{ for } i = 1, 2 \cdots n. \tag{4.13}$$

If we want to make C as large as possible under the conditions (4.13) we consequently have to put

$$C = \text{the smallest of the numbers } \frac{a_{i0}}{P_i} \qquad (i = 1, 2 \cdots n). \tag{4.14}$$

The numbers $\frac{a_{i0}}{P_i}$ we shall call the *absolute limitation numbers;* they indicate a boundary which the total volume of transactions of the community cannot surpass as long as none of the partakers shall be impelled to participate with a larger amount than he himself originally proposed, and the adaptation is made only by the principle of partakers' percentages.

Since a_{i0} and P_i are uniquely determined by the elements of the original request-matrix a_{ij}, C as defined by (4.14) is also determined by the elements of the request-matrix and consequently by (4.12) all the c_i. The partakers' percentages z_i may finally be determined by (4.2).

As an example let us consider the request-matrix given in (4.15).

(4.15)	Wants to buy from			Total
	Shoemaker	Tailor	Farmer	
Shoemaker	0	75	150	225
Tailor	80	0	360	440
Farmer	330	510	0	840
Total	410	585	510	1505

The matrix $(\alpha_{ki} - e_{ki})$ is in this case

(4.16)	$i=1$	2	3	Sum check
$k=1$	−1.00000	0.33333	0.66667	0
2	0.18182	−1.00000	0.81818	0
3	0.39286	0.60714	−1.00000	0

The adjoint of this is

(4.17)	$i=1$	2	3	
$k=1$	0.50325	0.73809	0.93939	(Check: all
2	0.50325	0.73809	0.93939	rows *equal*)
3	0.50325	0.73809	0.93939	

and consequently the relative volume numbers

$$\begin{aligned} P_1 &= 0.23077 \\ P_2 &= 0.33846 \\ P_3 &= 0.43077. \\ \text{Total } P_0 &= 1.00000. \end{aligned} \tag{4.18}$$

The limitation numbers are consequently

$$\begin{aligned} \frac{a_{10}}{P_1} &= 975 \\ \frac{a_{20}}{P_2} &= 1300 \\ \frac{a_{30}}{P_3} &= 1950. \end{aligned} \tag{4.19}$$

Hence partaker No. 1 (the shoemaker) is the minimum factor, and the net total volume of the operations for the whole community will be

$$C = 975. \tag{4.20}$$

The corrected sales of the individual partakers are thus by (4.12)

$$\begin{aligned} c_1 &= 225 \\ c_2 &= 330 \\ c_3 &= 420 \\ \text{Total } & 975. \end{aligned} \tag{4.21}$$

which by (4.2) corresponds to the following partakers' percentages

$$\begin{aligned} 1 + z_1 &= 1.000 \\ 1 + z_2 &= 0.750 \\ 1 + z_3 &= 0.500. \end{aligned} \tag{4.22}$$

Using these percentages we finally get the corrected exchange-matrix expressed in (4.23).

(4.23)	Is allowed to buy from			Total
	Shoemaker	Tailor	Farmer	
Shoemaker	0	75	150	225
Tailor	60	0	270	330
Farmer	165	255	0	420
Total	225	330	420	975

5. Generalized Principles of Correcting the Request-Matrix

The analysis of the last Section shows that if each person's *relative* distribution of purchases shall be strictly maintained, it may be necessary to reduce the *total* volume of transactions for the community considerably in order to find a complete demand-supply balance. We shall now discuss certain generalizations of the correction principles that may allow a better utilization of the productive capacity of the partakers.

Instead of using a set of reduction percentages z_i to be applied to each person, the idea may present itself to use a set of percentages x_j for each commodity: these percentages would then be a sort of *rationing-percentages*, it being understood that the rationing may be either positive or negative.

If we do not require the demand-supply balance to be fulfilled, we can, of course, always, by using an appropriate system of rationing-coefficients, assure full utilization of the capacity of the parties. We can simply let everybody produce to capacity and then determine the distribution by means of the rationing-percentages. If the demand-supply balance shall be fulfilled for each person, such an arrangement is not, however, possible. In this case no real advantage would be gained by using rationing-percentages instead of partakers' percentages. Indeed, in such a closed system everybody is at the same time a supplier and a consumer, so that the rationing of the consumption would at the same time be a rationing of the supply. Working the problem through mathematically, we would be led to exactly the same kind of solution as we discussed in Section 4, the only difference being that we would now have interchanged rows and columns of the request-matrix.

If a real advantage is to be gained we must use *both* partakers' percentages and rationing-percentages. This will introduce new degrees of freedom in the system, and open up the possibility of a better adaptation.

It may be a question whether one should combine the two percentages additively or multiplicatively. This is largely a question of ease in computation; indeed, if none of the percentages is very large, the additive composition can simply be looked upon as an approximation to the multiplicative. In the following we will use the additive composition, so that the elements of the corrected matrix become

$$(5.1) \qquad (1 + z_i + x_j)a_{ij}.$$

In general we will not *a priori* make any assumption *about the sign* of the percentages z_i or x_j. The only thing we need to assume is that for any i, j $1+z_i+x_j$ is non-negative. For certain types of partakers an adjustment such as the one here considered will be more desirable than an adjustment strictly by partakers' percentages. For an ordinary consumer, for instance, it will in general be more advantageous to keep up as far as possible the total volume of his purchases than to maintain exactly the *relative* distribution over the various items in his budget.

If there is a great surplus of one of the commodities, that is, if there is in the request-matrix a big supply and a small demand, it would be necessary to make the corresponding rationing percentage x_j *very large* in order to assure even a reasonable utilization of the productive capacity for this commodity. But, on the other hand, it may be that the commodity in question takes only a small part of the total production capacity of the community, so that it would hardly be perceptible if the surplus of this commodity—or at least a part of it—were distributed over the *whole* group of partakers. This leads to introducing a third type of coefficiently y_j, to be applied to the *total* request of each person. In other words the elements of the corrected matrix would be

$$(5.2) \qquad (1 + z_i + x_j)a_{ij} + a_{i0}y_j.$$

The percentages y_j must be assumed non-negative, otherwise it may happen that some of the elements in the corrected matrix become negative. For instance, if person No. 1 did not request the commodity j at all, but did express a desire to take some part in the exchange activity, the corrected element (5.2) would become negative if y_j was negative. The percentages y_j are thus essentially suited only to dispose of an existing *surplus*. They may, therefore, be called the *surplus coefficients*.

The percentages y_j have the effect of adjusting the surplus (or part of it) according to the *total* purchases proposed by each person. No regard is taken to the relative composition of the person's request. For instance, if a person has made a total request for \$2000, and it is decided to dispose of a butter surplus by putting the surplus coefficient for butter equal to $0.01=1$ per cent, then the partaker in question would receive a warrant on butter for \$20 (in addition to what he

may himself have requested of butter). If the partaker is, say, a shoe factory whose original request only involved certain factors of production like hide, labor hours of a certain category, etc., it would have no immediate use for the butter received. Of course, there would always be possibilities of utilizing it in a further indirect exchange, but that is a different question. The organized exchange should of course aim at producing the best possible harmony right from the beginning. It may, therefore, be found expedient to discriminate between various types of partakers, and let these types be affected differently by the surplus percentages. The simplest way in which this can be obtained seems to be to make the discrimination according to the relative composition of the purchases as it is displayed in the original requests from the various partakers. In other words, instead of the term $a_{i0}y_j$ in (5.2) one would introduce the weighted sum

$$\sum_k a_{ik}y_{kj} \tag{5.3}$$

where y_{kj} is a set of percentages. If these percentages are independent of k, i.e., if we have $y_{kj}=y_j$, we get back to the case expressed by the last term in (5.2).

If the last term in (5.2) is replaced by an expression like (5.3), it will from the formal point of view be unnecessary to retain the term $x_j\, a_{ij}$. Indeed, the effect of this term is represented by the term $k=j$ in (5.3). If this interpretation is adopted, the matrix y_{kj} must be assumed to consist exclusively of non-negative elements, except possibly the diagonal elements that may be negative. For convenience in handling the formulae it will, however, be better to retain also the term with x_j. If this is done, it does not restrict generality to assume that the matrix y_{kj} has all its diagonal elements equal to any set of given numbers, for instance zeros. The term with z_i in (5.2) must under any cirsumstances be retained, as its effect is not in general taken account of by (5.3).

We are thus led to considering the following form of the elements of the corrected matrix.

$$(1 + z_i + x_j)a_{ij} + \sum_k a_{ik}y_{kj}. \tag{5.4}$$

Finally it may be desirable to introduce a generalization with regard to the demand-supply balance. To explain the reason for this, let us imagine that one of the parties involved would attempt to make an extra profit by raising the price of its product and at the same time trying to force its sale through the exchange organization. Such a situation is quite conceivable if the correcting of the request matrix is made with a view to utilizing as far as possible the supply from the partakers as expressed by the request-matrix. To meet this situation it is

desirable to introduce into the planned circulation mechanism some element that may be manipulated so as to produce an effect similar to the one which would in a free market be produced by the adjustment of relative prices. This can be obtained simply by deciding that in the correcting of the request-matrix there shall *not* be arranged for an exact equality between the value of the goods delivered and those received by each partaker, but that the former value should *exceed*, or possibly be short of, the latter by a certain amount. The difference would—if it is positive—appear as a *service tax* levied by the exchange organization. In the general theoretical set-up it will be desirable to introduce such a parameter for each partaker. This means that the demand-supply balance would now be written in the form

$$(5.5)\quad \begin{cases} c_i = \text{sum of row No. 1 in the corrected matrix} \\ c_i + \rho_i = \text{sum of column No. 1 in the corrected matrix} \end{cases}$$

where ρ_i is the service tax levied on partaker No. 1.

Since the sum of the rows must be equal to the sum of the columns, we must have

$$(5.6)\qquad \rho_0 = \sum_k \rho_k = 0.$$

This means that some of these taxes must be positive and some negative. To avoid this limitation and make the scheme perfectly general, we only have now to let one of the partakers be the organization itself. This would establish a close accounting system where the condition (5.6) is fulfilled, but where no assumption is made on the service parameters ρ_i for the real partakers.

The parameters ρ_i may be used for various purposes. Besides the one already mentioned, they may be used in an attempt to counteract the disastrous effect on the whole exchange activity of the community that is occasionally produced by price changes that deprive certain classes of producers of the greater part of their purchasing power. The service parameter for this group of partakers would then have to be made negative.

The service parameters may also be utilized to allocate to the partakers the administration costs of the system in case it is found desirable that the costs should be covered by the system itself and not, for instance, by federal or state funds. In this respect the service parameters would play exactly the same rôle as the surcharge made to the net actuarial premium in the life insurance companies.

If the exchange system becomes widely used, it would also, through the technique of the service parameters, be possible to realize various forms of communication between the exchange system and the state

or federal budgets. Conceivably the service parameters may even be used for regular fiscal purposes.

6. The Limitational Condition for the Total Volume of Operations

How shall these various parameters be fixed in order to assure the "best possible" adaptation of the system to the desires of the partakers? At a first glance the great number of parameters and their very different nature seems to make it nearly impossible to keep track of the various effects and to recognize clearly the actions and counteractions. By a suitable grouping of the questions that arise, and by making a few—as it seems, rather plausible—assumptions, it is, however, possible to bring order into the matter and work out a technique amenable to numerical computation.

We shall assume that the application of the partakers' percentages ε_i does not give rise to any optimum consideration. The general line of analysis will therefore be to start by investigating what the total volume of operation of the community C will be if all the *other* parameters are *given* and the z_i are simply adapted so as to realize the largest possible volume compatible with the condition that none of the originally proposed capacities a_{i0} shall be surpassed. The total volume C as determined by this procedure will be a function of the other parameters, that is of the x, y, and ρ. The next step will then be to make this function C as large as possible. This must be looked upon as involving a certain "cost" which is the *inconvenience* created by forcing any of the variables in C, namely the x, y, and ρ, out of their "natural" position, which is zero. These two considerations, that of increasing the total volume C and that of keeping down the inconvenience must now be weighted against each other.

Logically this problem is very much like the general problem we encounter in productivity theory: C may be considered as the "product" and the x, y, and ρ, the "factors of production." As in productivity theory, it will also here be found expedient to separate the problem of adaptation in two parts: First, the "substitution" by which the minimum cost is obtained which corresponds to a given volume, then the "volume adaptation," which involves a study of how total volume varies as a function of total cost along the optimum curve. This optimum curve (or the "substitumal" as I call it in my lectures on productivity) is defined as the locus of points in factor space where to any given magnitude of the product the substitution is completely realized. Much of the technique of this productivity analysis can be applied to the present case, although some aspects of the problem now are more complicated than it is customary to consider it in productivity theory.

The first step in the analysis of the adaptation process, as here out-

lined, is to eliminate the partakers' percentages. Taking the sum over j and the sum over i respectively in (5.4) and using the balancing principle expressed in (5.5) we get the following two fundamental equations.

$$(6.1) \qquad (1 + z_i)a_{i0} + \sum_k a_{ik}x_k + \sum_k a_{ik}y_{k0} = c_i$$

$$(6.2) \qquad a_{0j} + \sum_k z_k a_{kj} + a_{0j}x_j + \sum_k a_{0k}y_{kj} = c_j + \rho_j.$$

In the second of these equations we change j to i, and insert the expression for the z's taken from the first equation. This gives

$$a_{0i} + \sum_k \left\{ c_k - a_{k0} - \sum_h a_{kh}(x_h + y_{h0}) \right\} \alpha_{ki} + a_{0i}x_i + \sum_h a_{0h}y_{hi} = c_i + \rho_i$$

which reduces to

$$(6.3) \qquad \sum_k c_k(\alpha_{ki} - e_{ki}) = r_i$$

where

$$(6.4) \qquad r_i = \rho_i + \sum_k (\gamma_{ik} - e_{ik})a_{0k}x_k + \sum_k (\gamma_{ik}y_{k0} - y_{ki})a_0$$

and

$$(6.5) \qquad \gamma_{ij} = \sum_k \alpha_{ki}\beta_{kj}$$

$$(6.6) \qquad \beta_{ij} = \frac{a_{ij}}{a_{0j}}.$$

The matrices β and γ are determined by the data in the original request-matrix.

If it is desired to avoid the row sums y_{k0} in (6.4) and express the formula only by means of the elements of the matrix y_{ki}, the last term of formula (6.4) becomes a double sum, we get

$$(6.7) \qquad r_i = \rho_i + \sum_k (\gamma_{ik} - e_{ik})a_{0k}x_k + \sum_{hk} (\gamma_{ik} - e_{ik})a_{0k}y_{kh}.$$

The matrices β and γ have obviously the properties

$$(6.8) \qquad \beta_{0j} = \sum_k \beta_{kj} = 1$$

$$(6.9) \qquad \gamma_{0j} = \sum_k \gamma_{kj} = 1$$

which, since $\rho_0=0$, shows that, quite regardless of what values are selected for the parameters x and y, we must always have

$$r_0 = \sum_i r_i = 0. \tag{6.10}$$

From (6.10) it follows that the system (6.3) always has a solution. Indeed the matrix of the coefficients has by (4.5) the property that the sum in any row is zero. The determinant of the coefficients is therefore zero, and it will not (except by coincidence) be of lower rank. But any matrix obtained from $(\alpha_{ni}-e_{ni})$ by replacing one of the rows by the numbers $r_1 \cdots r_n$ will by (6.10) have exactly the same property, which shows that (6.3) is a singular system of compatible equations.

The coefficients in the left member of (6.3) depend only on the elements of the originally given request-matrix. *The effect of the parameters x, y, and ρ, is thus completely incorporated in the quantities r_i that form the right member of* of (6.3). Furthermore, we note that it is only in this second member that the present equation distinguishes it-

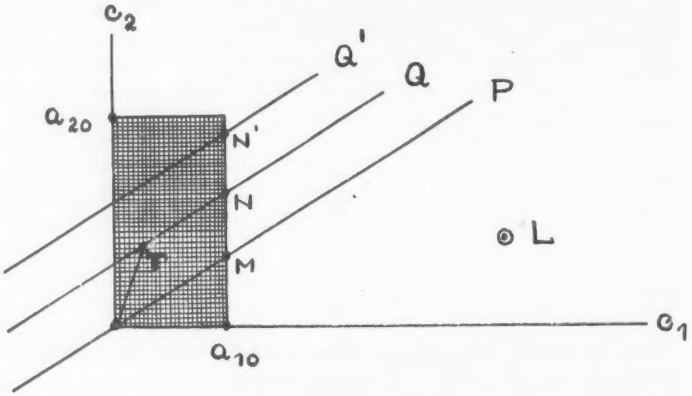


FIGURE 1

self from (4.7), (6.3) is simply the inhomogeneous equation corresponding to the homogeneous equation (4.7). It is, therefore, to be expected that some of the essential features of the simpler case discussed in Section 4 will be retrieved in the present case.

The fact that the effect of all the parameters x, y, and ρ, is incorporated in the r's makes the further line of analysis clear: We must determine what the maximum volume of transactions will be if the r's are considered as a single set of *given* parameters.

We shall first illustrate the matter graphically.

A system of two equations in the two unknowns c_1, c_2 is completely determinate when the matrix of the coefficients is non-singular. In this case the solution is represented by a *point* in (c_1, c_2) space, say the point L in Figure 1. If the matrix of the coefficients is singular (but not of a rank lower than 1) and the system is homogeneous, i.e., if the right member is equal to zero, there also always exists a solution, but this solution has one degree of freedom; more precisely it is represented by *a straight line* through the origin, say the line P in Figure 1. Finally, if the system is singular (but not of a rank lower than 1) and inhomogeneous with a right member that satisfies the compatibility conditions, the solution is represented by a straight line that does *not* go through origin, say the line Q in Figure 1. The *direction* of the line Q is just the same as the direction of the line P that represents the solution of the corresponding homogeneous equation. All possible systems with the

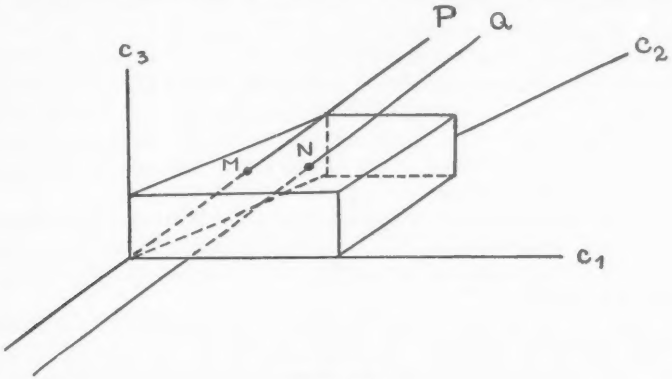


FIGURE 2

same matrix of coefficients in the left member will thus have the same basis direction, and only differ with regard to a parallel displacement, a *translation* of the basis line, this translation depending on the right member of the system of equations. It seems natural, therefore, in this case to look upon the solution as being made up of two components: first a *translation* T from origin to some point on the line Q (see Figure 1) and then a movement *along* Q.

A similar interpretation applies in the case of three equations with three unknowns. If the three-rowed matrix of the coefficients is non-singular, the solution is represented by a *point* in (c_1, c_2, c_3) space. If the matrix is of rank 2, and the system homogeneous, i.e., without second member, the solution is a straight line through origin (see P in Figure 2). Finally, if the matrix is of rank 2 and the system inhomo-

geneous with a right member satisfying the compatibility conditions, the solution is a straight line that does *not* go through origin (see Q in Figure 2).

(6.3) is an example of a system whose solution will be represented by a straight line as indicated in Figure 1 and Figure 2. If the r's are all zero, the straight line in question will go through origin, otherwise the solution contains a translation from origin in addition to a movement along the line.

These diagrams illustrate in a striking fashion what is obtained by introducing the parameters x, y, and ρ, in addition to the partakers' percentages. If only the latter are used, we have to move on the line P, which is independent of the x, y, and ρ, and uniquely determined by the original request-matrix. In Figure 1 this gives the point of exit M, where the total capacity of No. 1 is used, but only a small fraction of the capacity of partaker No. 2. The introduction of the parameters x, y, and ρ, *defines the translation* T, and by going on the new line Q we reach a point N where a much larger portion of the capacity of No. 2 is used. A still larger translation would give the line Q' with the point of exit N', where nearly the full capacity of No. 2 is used. The *direction* of the line Q is given by the elements of the request-matrix and cannot be altered, but a *translation* may be produced by using a set of values of the adaptation parameters x, y, and ρ.

A similar interpretation applies in the case of three variables. Figure 2 exhibits, for instance, a situation where the points M and N are such that only No. 3 is utilized to capacity, while both No. 1 and No. 2 have unused capacity.

Now let us express this algebraically. The equation (6.3) is of the form (22.5). Consequently an application of the formula (22.21) for $A_{ki}=\alpha_{ki}$ and $s_i=r_i$ would in point of principle furnish a solution of our system. In practice, however, it is convenient to make a slight modification which will greatly increase the rapidity with which the iteration process of Section 22 converges. To see the nature of this procedure, let us put $A_{ki}=\alpha_{ki}$ and $s_i=0$ in (22.6), and let us introduce a tentative set of numbers for the c's in the left member. By so doing we shall have determined a *new* set of c's, and this new set may be taken as a second approximation to the correct c's, that satisfy the homogeneous equation obtained from (22.5) by equating the s's to zero. Incidentally, the equation (22.11) shows that the *sum* of the c's in the second approximation is the same as in the first, which furnishes a convenient check on the computation. From the second approximations we may determine a third, and so on.

By using this process on a matrix A_{ki} that has the usual properties of a relative request-matrix α_{ki}, one will find that the approximations

for, say, c_i fall alternately on the upper and lower side of the true value; both series approaching about equally rapidly to the true value. This suggests taking always two steps at a time and using the average value of c_i obtained at the two steps. Instead of doing this numerically we may determine quite generally the formula of the new iteration process that thus arises. Doing this we find that the new process corresponds to putting

$$A_{ki} = \frac{\alpha_{ki} + e_{ki}}{2} \tag{6.11}$$

$$s_i = \tfrac{1}{2} r_i \tag{6.12}$$

and then using this new matrix A_{ki} in the formulae (22.9), (22.18) and (22.17) that serve to determine the matrix Q_{ki}.

That this method will lead to a determination of the c's also in case of the inhomogeneous equation can be verified directly. From (6.3) we get indeed immediately by adding and subtracting e_{ki} in the parentheses under the summation sign

$$\sum_k c_k(\alpha_{ki} + e_{ki} - 2e_{ki}) = r_i$$

that is

$$\sum_k c_k\left(\frac{\alpha_{ki} + e_{ki}}{2} - e_{ki}\right) = \frac{1}{2} r_i .$$

This is nothing but equation (22.5) if we put A_{ki} and s_i equal to the expressions (6.11) and (6.12) respectively.

The solution c_i of (6.3) can consequently be written in the form

$$c_i = CP_i + \tfrac{1}{2} \sum_k r_k Q_{ki} \tag{6.13}$$

where Q_{ki} is the matrix (22.20), A_{ki} being defined by (6.11). The numbers P_i are determined by (22.14).

Incidentally, the numbers P_i in (6.13) are identical with the numbers P_i defined by (4.9). Indeed, by putting all the r's equal to zero in (6.13) we see that the P_i that occurs in (6.13) must be proportional to the elements in a row of the adjoint of $(A_{ki} - e_{ki})$. But we have

$$A_{ki} - e_{ki} = \tfrac{1}{2}(\alpha_{ki} - e_{ki}) \tag{6.14}$$

so that the elements of a row in the adjoint of $(A_{ki} - e_{ki})$ are proportional to elements in a row of the adjoint of $(\alpha_{ki} - e_{ki})$, i.e., proportional to the numbers P_i defined by (4.9). Since the sum of the P_i in (6.13) is equal to the sum of those defined by (4.9), namely 1, the two sets must be identical.

For brevity let us put

$$u_i = -\frac{1}{2P_i}\sum_k r_k Q_{ki}. \tag{6.15}$$

The numbers u_i are going to play a fundamental rôle in the following. They satisfy the identity

$$\sum_i u_i P_i = 0. \tag{6.16}$$

This follows by multiplying (6.15) by P_i and performing a summation over i, using (22.19).

In terms of the u_i formula (6.13) may be written

$$c_i = (C - u_i)P_i. \tag{6.17}$$

Performing here a summation over i we see by (4.10) and (6.16) that equation (4.11) is satisfied also in the present general case.

If a set of parameters x, y, and ρ, are given, the numbers u_i are determined. This means by (6.17) that we have determined the straight line on which the point $(c_1 \cdots c_n)$ is bound to lie. The parameter C that represents the movement along the line is, however, still undetermined. It is finally fixed by requiring that none of the c_i shall be larger than the corresponding a_{i0}. That is to say, we must have

$$(C - u_i)P_i \leqq a_{i0} \text{ for all } i$$

which is equivalent to

$$C \leqq \frac{a_{i0}}{P_i} + u_i \text{ for all } i. \tag{6.18}$$

The largest C that satisfies the condition (6.18) is obviously

$$C = \text{smallest of the numbers } C_1\, C_2 \cdots C_n \tag{6.19}$$

where

$$C_i = \frac{a_{i0}}{P_i} + u_i. \tag{6.20}$$

The first term in the right member of (6.20) is independent of the parameters x, y, and ρ, and depends only on the elements of the original request matrix.

Since also in the present general case, C designates the total volume of operations for the community, the preceding formulae may be given the following interesting interpretation:

If the total volume of operation for the community shall be brought up

to a given size C, it is necessary and sufficient that there be fixed a set of values of the parameters $u_1, u_2 \cdots u_n$ *such that the smallest of the numbers* (6.20) *becomes not less than C. This we shall call the limitational condition.*

Suppose for instance that we have four commodities (or services) and that the magnitudes a_{i0}/P_i, as determined from the elements of the original request-matrix, turn out to be

i	$\frac{a_{i0}}{P_i}$
1	1300
2	975
3	625
4	800.

Then, if only partakers' percentages are used, the total volume of transactions for the community would be 625. If a larger total volume will be possible, say a volume of 650, it is necessary to have $u_3 \geqq 25$. If we would arrange for a total volume of 825, it would be necessary to have $u_3 \geqq 200$ and $u_4 \geqq 25$, and so on.

The parameters u_i may thus be looked upon as a set of limitational factors that concur in determining the size of C. For any given set of values of the u's one or more of them will be *minimum factors*. The situation is exactly as for limitational factors in productivity theory. A lower limit for each u_i is determined as a function of C. The expression "limitational factors" or simply "factors" will frequently be used in the following to designate the u's.

The fixation of the factors $u_1, u_2 \cdots u_n$ involves a cost element, namely the inconvenience attached to giving x, y, and ρ, such magnitudes as will produce the set of u's considered. How can the parameters x, y, and ρ, be determined so as to produce the necessary set of u's with the least possible inconvenience? This will be the topic of the next Section.

7. *A General Differential Method of Adaptation for the Minimum Factors*

A given set of the u's can be produced by many different choices of the parameters x, y, and ρ. This selection may also be described in productivity terms, but now the u's have to be considered as the things produced and the x, y, and ρ, the factors of production; we may perhaps call them the *elementary* factors as distinct from the u's that are factors in the production of C.

In considering the relations between the u's and the elementary factors, we have essentially a problem in *joint* production: There are n "products" $u_1 \cdots u_n$ being produced simultaneously by the use of a number of common factors. The principle of substitution leading to the minimum cost combination is in this case much more complex than the corresponding principle for a single commodity production, of the type usually considered in productivity theory. But, on the other hand, the form of the functional relationship between products and elementary factors is simpler in the present case. It is indeed a linear relationship. By (6.15) the u's are expressed linearly in terms of the r's, and by (6.7) the r's are expressed linearly in terms of the x, y, and ρ; it would not be difficult, therefore, to write out the general form of the relations connecting the u's with the elementary factors.

In the following we shall not, however, use this general formula. For simplicity we shall carry the rest of the discussion through on the assumption that $\rho_i=0$, and that $y_{kh}=y_h=$independent of k. This means that we consider the case where the corrected matrix is of the form (5.2). In this case the expression (6.7) reduces to

$$r_i = \sum_k(\gamma_{ik} - e_{ik})a_{0k}x_k + y_0a_{0i} - y_ia_{00}. \tag{7.1}$$

Inserting this into (6.15) we get

$$u_i = \sum_k f_{ik}'x_k + \sum_k f_{ik}''y_k \tag{7.2}$$

where

$$f_{ik}' = -\frac{a_{0k}}{2P_i}\sum_h(\gamma_{hk} - e_{hk})Q_{hi} \tag{7.3}$$

$$f_{ik}'' = -\frac{1}{2P_i}\sum_h(a_{0h} - e_{hk}a_{00})Q_{hi} = \frac{1}{2P_i}\left\{a_{00}Q_{ki} - \sum_h a_{0h}Q_{hi}\right\}. \tag{7.4}$$

The matrices (7.3) and (7.4) satisfy the identities

$$\sum_i P_i f_{ik}' = 0 \qquad \sum_i P_i f_{ik}'' = 0 \text{ for all } k. \tag{7.5}$$

By (7.2) u_i is defined as a linear form in the $2n$ variables $x_1, x_2 \cdots x_n$, $y_1, y_2 \cdots y_n$. For convenience in handling the formulae, it is better to denote these variables consecutively

$$x_1, x_2 \cdots x_N \tag{7.6}$$

where $N=2n$ and $x_K=y_{K-n}$ when $K>n$. With this notation we may write

$$u_i = \sum_K f_{iK}x_K \tag{7.7}$$

where K runs from 1 to N, and f_{iK}, is the n rowed and N columned matrix obtained by joining f_{ik}'' on to the right side of f_{ik}'. The matrix f_{ik} obviously satisfies the relation

$$\sum_i P_i f_{iK} = 0 \text{ for all } K. \tag{7.8}$$

All the elements of the matrix f_{ik} are by the preceding formulae determined in terms of the elements of the original request-matrix a_{ij}.

At this stage of the analysis we have to introduce an assumption about the nature of the inconvenience connected with a given deviation of the percentages $x_1 \cdots x_N$ from zero.

Suppose for a moment that we had succeeded in defining numerically an index function $\Omega(x_1 \cdots x_N)$ that could roughly be taken as indicating the total inconvenience caused by letting the parameters x_K assume given magnitudes. The nature of the variation of the function Ω could, for instance, be indicated by means of tables or graphs, or perhaps in part by analytical expressions with given numerical parameters. The function Ω being thus defined, we could also by a numerical differentiation process indicate roughly the value of the marginal inconveniences

$$\Omega_K = \frac{\partial \Omega}{\partial x_K} \tag{7.9}$$

in the various positions. Possibly we should be obliged in certain points to distinguish between forward and backward derivatives. This would in particular be the case for $\Omega_{n+1} \cdots \Omega_N$ in points where any of the $x_{n+1} \cdots x_N$ is zero. Indeed, we assume that all the $x_{n+1} \cdots x_N$ are non-negative, which can be interpreted by saying that an infinite inconvenience would be caused if any of the $x_{n+1} \cdots x_N$ passed from a non-negative to a negative value.

Instead of assuming Ω to be numerically given we could also take the Ω_K as the primary data; that is to say, by tables, graphs, or otherwise, we could attempt to indicate the intensity of *increase* in total convenience that would follow if, from a given point x_K we would increase one of the parameters x_K. From the theoretical viewpoint this would be a more general definition of the inconvenience, and *in practice it would probably be easier to formulate it this way.*

I believe that it would not be beyond practical possibilities actually to carry through a numerical definition of Ω or the Ω_K in a given exchange system. Suppose, for instance, that it is decided to work only with the rationing coefficients $x_1 \cdots x_n$. These may be either positive or negative. As a first approximation it seems fair to assume that a small positive value of any of them would be about as undesirable as

a small negative value (the total purchase of each partaker being in any case adjusted to his total sales). As a first approximation we may further assume the marginal inconveniences to be independent. This suggests representing the part of the total inconvenience that is due to a given value of x_K by a parabola around origin, in other words by a function of the form

$$\frac{x_K^2}{2\epsilon_K} \tag{7.10}$$

where ϵ_K is a constant. The shape of such a curve is indicated in Figure 1.

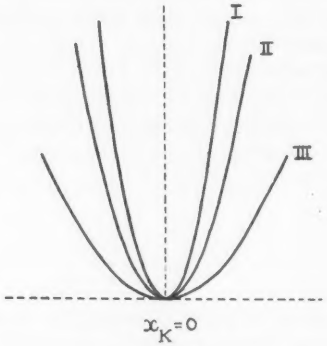


FIGURE 1

A small value of the constant ϵ_K means that the inconvenience increases very rapidly if x_K deviates from zero, (see curve I in Figure 1) while a small ϵ_k means a small increase in inconvenience. Thus the size of ϵ_K will be characteristic for the commodity (or service) in question. It does not seem to be beyond practical possibilities to fix roughly the value of ϵ_K for some of the main commodities that would be likely to enter into the exchange system. For the parameters $y_1 \cdots y_n$ the marginal inconvenience curves would be asymmetric, shooting up to infinity on the negative side.

Assuming the functions Ω_K to be numerically given, let us now consider the substitution problem, that is to say, the problem of how the parameters x_K are to be adapted so as to realize the smallest cost or inconvenience that is possible when it is given that the total volume of operation C shall have a given size.

If C shall have a given magnitude, it follows from the limitational condition of Section 6 that there are certain of the factors u_i that must necessarily have at least certain given magnitudes. This limitational

effect of the factors u we shall now consider a little more closely, and particularly in relation to a given point in x_K space.

If a point in x_K space is given, then all the factors $u_1 \cdots u_n$ are given by (7.7) and consequently also the total volume of transaction which it is possible to realize in this point. Those factors for which a_{i0}/P_i+u_i is just equal to this total volume are the minimum factors in the point x_K considered; we may also say that these factors are *minimal* in this point.

Now suppose that in addition to a given point x_K we consider some *given* value C of the total volume, which may be equal to or different from the maximum volume belonging to the point x_K. Each factor u_i which in the point x_K is less than what is needed to produce C we shall call a *deficiency factor* for C in the point x_K, or, shorter, we may say that this factor is *deficient* for C in the point x_K. Those factors for which $a_{i0}/P_i+u_i=C$ may be called *barely sufficient* for C in the point x_K, and those for which $a_{i0}/P_i+u_i>C$ *plentiful* for C in the point x_K.

If it is desired to realize a certain given value C of the total volume, each point in x_K space may thus be classified according to its properties with regard to the given value of C. This means that the space x_K is divided into an extremely complicated system of compartments, partly overlapping and partly containing each other, and such that in each compartment a definite set of factors (perhaps all) is deficient for C, some other set is barely sufficient, and others plentiful, for C.

To seek the minimum of the total inconvenience function Ω in these circumstances will, therefore, involve such a mixture of problems in continuous variation and in multidimensional boundary conditions that, *prima facie*, it seems impossible to keep track of all the various alternatives.

There exists, however, a way out which I believe will be applicable even in the case of a large number of variables. It is quite impossible to consider the x_K space in *its totality* and by *a priori* considerations to pick out that particular point where the minimum of Ω is realized. It is impossible here to proceed by the same methods which are—alas, so superficially—used in general economic equilibrium theory, namely to determine the minimum (or maximum) of an index function of several variables just by equating its partial derivatives to zero.

In the present case it will not, indeed, be possible to study the x_K space *at large*, we must be satisfied by studying it *differentially*, but this will furnish a solution that satisfies most of the practical requirement. *It means that we have to seek the solution of the adaptation process by small tentative steps.*

Following strictly the theoretical distinction between the substitution process and the volume adaptation, we should then have to con-

sider first a tentative movement by small steps during which we keep all the time the total volume constant, while constantly reducing the total inconvenience. Having achieved this we may start to change the volume. In practice it will be found better to have both objects in view during each step. In other words, if we start tentatively in a given point x_K (for instance, where all the x_K are zero), we may decide to make a movement such that total volume changes *a certain amount* which we fix tentatively. This is the form of movement we shall consider in the following. It contains of course as a special case the movement where total volume is kept constant.

Suppose then that we start in some given point x_K where the total volume turns out to be C. We decide that we want to move to some neighboring point

$$x_K' = x_K + \delta x_K \tag{7.11}$$

where total volume is

$$C' = C + \delta C \tag{7.12}$$

δC we look upon as given, and the quantities δx_K we consider as unknowns to be determined in such a way that the total inconvenience Ω is decreased as much as possible (or increased as little as possible) by the passage from x_K to x_K'.

If we make a small step from the starting point x_K to some point x_K' not imposing any other condition on the movement than the minimalization of Ω, it may conceivably happen that, in the new point, the total volume actually turns out to have the prescribed magnitude (7.12). But in most cases that will not, of course, be so. With the shape of the function Ω that is appropriate to the optimum problem in circulation planning, the cost minimalization will—if not checked by side relations—*have a tendency to lower the u's;* and this may have the effect that some or more of the factors $u_1 \cdots u_n$ will in the new point turn out to be deficiency factors for the prescribed magnitude $C+\delta C$.

It will therefore be necessary to pick out at least a certain group of factors, say the m factors

$$u_p, u_q \cdots u_t \tag{7.13}$$

and impose on the movement from x_K to x_K' the side condition that these factors shall *not* be deficiency factors in the new point.

Conceivably the manner in which the total inconvenience depends on the various parameters may be such that, for one or more of the factors u, it will be easier to arrange for a higher value than for a smaller; it might for instance be easier to make u_3 equal to 30 than to 25. In practice such cases will, however, be very exceptional. It therefore

seems plausible to replace the condition that a given factor shall not be deficient for $C+\delta C$, in the new point, with the condition that it shall be barely sufficient. In other words the side condition for the factors (7.13) will be that

$$\sum_K f_{iK} x_K' = u_i' \qquad (i = p, q \cdots t) \tag{7.14}$$

where u_p', $u_q' \cdots u_t'$ are the magnitudes of the factors considered that will barely be sufficient for the prescribed volume $C'=C+\delta C$. By (6.19) these magnitudes are

$$u_i' = C' - \frac{a_{i0}}{P_i}. \tag{7.15}$$

Introducing the increments instead of the total values we can write (7.14) in the form

$$\sum_{K=1,2\cdots N} f_{iK} \cdot \delta x_K = \delta u_i \qquad (i = p, q \cdots t) \tag{7.16}$$

where

$$\delta u_i = u_i' - u_i \tag{7.17}$$

u_i being the value of the factors in the starting point x_K.

The selection of the group of factors, for which the non-deficiency condition (7.16) is imposed, must in practice to some extent be based on a rough estimate of the kind of variations in the u_i that are likely to accompany a given change in the x_K. It is always desirable, of course to make the group as small as possible, because the smaller the number of side conditions, the larger the possibility of obtaining a large net reduction in Ω. It will therefore be desirable to make a first attempt with a rather small group of conditional factors, and then little by little to increase the groups, if it is found that some of the free factors actually turn out to become deficiency factors in the new point.

Since the number of variables is finite, there is a finite number of possibilities to consider, a complete solution of the problem is therefore always possible in principle, but in practice there can never, of course, be any question of trying more than a very few of those combinations that seem really important. For instance, if the step considered involves a *non-decrease* in total volume, i.e., if δC in (7.12) is non-negative, it will probably be practical first to make a step where one includes in the conditional factor group *those factors that are deficient or barely sufficient for* $C+\delta C$ *in the starting point* x_K, and possibly some others that are very close to having this property, i.e., for which the difference $C+\delta C-(a_{i0}/P_i+u_i)$ is very small in the starting point.

So much for the side conditions. Now let us consider the minimalization process itself.

If the marginal inconveniences Ω_K change in a continuous way in the vicinity of the starting point x_K, the change in total inconvenience by going from x_K to x_K' will be approximately

$$\sum_K \Omega_K \cdot \delta x_K \tag{7.18}$$

provided the δx_K are small.

If some of the Ω_K should change discontinuously in the vicinity of the point considered, for instance, become infinite for a backward movement, we may simply consider the corresponding δx_K as a constant equal to zero, and not as a variable. We may, therefore, always—by a proper interpretation of the δx_K—take the minimalization of (7.18) as the object of the adaptation.

This minimalization is to take place under the side condition (7.16). Since we have formulated the minimalization problem in the differential form (7.18), there is, however, also another kind of side condition to take into account; viz., that all the δx_K shall be "small." This condition will in practice have to be interpreted *cum grano salis*, but it cannot simply be neglected.

The condition can be formulated by defining a closed surface around x_K, say a sphere, an ellipsoid, or a little "box," within which the point $x_K + \delta x_K$ shall lie. The form of the surface involves a definition of the *metric* of the space x_K, i.e., of the notions of direction and length. In point of principle, the x_K cannot be considered as comparable in the same sense as the coordinates in a physical space, so that the metric of the space x_K is not defined by any obvious *a priori* principle; but for practical purposes we may simply make use of the facts that all the x_K are here pure numbers (percentages) and that they will, furthermore, in our problem, be of the same order of magnitude. It seems therefore, plausible to use the notions of direction and length in the ordinary way and to define the surface as a small N dimensional sphere.

This being so, if *no* other side condition than that of the "smallness" of the δx_K were imposed on the movement, the cost minimalization would be obtained by going from x_K *along the gradient of* Ω, that is along the vector whose components are $(\Omega_1 \cdots \Omega_N)$, and in the negative direction. The length of the movement would be determined by the radius of the limiting sphere.

If some other side condition is imposed, for instance, that the movement shall lie in a given *plane*, then the cost minimalization would be realized by going along the *projection* of the gradient on to this plane. Quite generally, if the movement is bound to lie in a certain $N-m$

dimensional linear manifold, as the one defined by (7.16) (where the f_{iK} and the δu_i now must be looked upon as given), the cost minimalization would be realized by going a certain "small" distance in the negative direction of the projection of $(\Omega_1 \cdots \Omega_N)$ on to this $N-m$ dimensional manifold.

We therefore have to determine the projection of the vector

$$-\lambda\Omega_K \tag{7.19}$$

on the $N-m$ dimensional manifold defined by (7.16), λ being a "small" positive quantity.

The solution of this problem is given by the formulae of Section 23. Let us first assume that all the derivatives Ω_K can be looked upon as continuous in the vicinity of the point considered. In this case all the δx_K are free variables (apart from the conditions (7.16)). The solution of the problem is, therefore, obtained by putting $-\lambda\Omega_K$ for x_K and δu_i for u_i in (23.35). This gives

$$\delta x_H = \lambda \sum_{K=1,2\cdots N} (F_{HK} - e_{HK})\Omega_K + \sum_{k=p,q\cdots t} \delta u_k \cdot f^*_{kH} \tag{7.20}$$

where F_{HK} is the projection matrix for the m-rowed and N-columned matrix consisting of the rows $p, q \cdots t$ from the matrix f_{iK} in (7.7). Choosing tentatively some small value of λ in (7.20) means that we decide upon the "length" of the step to take from the initial point x_K. The length of this step has a certain connection with the selection of the set of variables to be included in the conditional group (7.13). The smaller the step, the smaller the probability that any of the free factors shall break through and become deficiency factors in the new point.

If, in the starting point, some of the $x_{n+1} \cdots x_N$ are zero, we must take care not to make a step that will decrease any of these parameters, but, on the other hand, we must leave the possibility open of *increasing* any of them if that should fit in with the object pursued. In this case the derivatives $\Omega_1 \cdots \Omega_n$ are continuous, while the $\Omega_{n+1} \cdots \Omega_N$ are not all continuous; as a matter of fact if $x_K = 0$, $K > n$, Ω_K should be looked upon as being defined only for a forward movement.

A practical way of handling this situation seems to be the following: First the formula (7.20) is used *as if* all the variables had been free (apart from (7.16)). In other words (7.20) is used as it stands. If this gives a negative value for some of the $\delta x_{n+1} \cdots \delta x_N$ for which the corresponding $x_{n+1} \cdots x_N$ is zero, the computation should be repeated, this time considering these variables as *constants* so that the equation (7.16) now take on the form

$$(7.21) \qquad \sum_{K=P,Q\cdots T} f_{iK}\cdot\delta x_K = \delta u_i \qquad (i = p, q\cdots t)$$

where δx_P, $\delta x_Q \cdots \delta x_T$ are all the $\delta x_1 \cdots \delta x_N$ except the increments for those of the magnitudes $x_{n+1} \cdots x_N$ that are now considered as constants. The problem is thus now to determine the projection of the vector $-\lambda\Omega_K(K=P, Q \cdots T)$ on to the manifold defined by (7.21). By (23.35) the solution of this problem is

$$(7.22) \qquad \delta x_H = \lambda \sum_{K=P,Q\cdots T} (F_{HK} - e_{HK})\Omega_K + \sum_{k=p,q\cdots} \delta u_k\cdot \overset{*}{f}_{kH}$$

where F_{HK} now is the projection matrix for the m-rowed and M-columned matrix formed by the rows $p, q \cdots t$ and the columns $P, Q \cdots T$ of the matrix f_{iK} of (7.7).

By the formulae (7.20) and (7.22) a series of tentative steps may be made, realizing more and more perfectly the minimalization of the total inconvenience, and increasing at the same time the total volume C little by little.

In each point that is successively passed during the computation, the magnitude Ω (or, more generally, the integral of the scalar product $\Sigma_K\Omega_K\cdot\delta x_K$ along the path followed) should be computed. The size of this quantity will be the guide by which to judge the feasibility of a further increase in C. If a point is reached where Ω (or the integral) has become so large that a further increase is deemed undesirable, the process should be stopped. The point x_K thus determined will then define the final corrected request-matrix.

It would perhaps be possible to combine the study of the increasing cost function Ω with the study of an increasing total utility function ϕ representing the "desirability" of an increase in the quantities consumed by the various parties. The difference $\phi-\Omega$ would then be an index function to be maximized. In this case the adaptation process itself may lead to a definite final maximum, without that amount of arbitrariness which in the preceding method was involved in judging when Ω had become so large as to make it undesirable to continue. I shall not, however, discuss this possibility any further here.

8. An Analytic Method of Adaptation

The method of adaptation described in the preceding Section is perfectly general in the sense that no assumption is made about the analytical form of the functions that define the total inconvenience Ω. The method of the preceding Section may indeed be applied even if Ω or the marginal inconveniences Ω_K are given only in the form of tables or graphs.

If certain assumptions are made about the analytical form of the

functions Ω or Ω_K, the work may in many respects be simplified, particularly because it may then be possible to indicate in explicit form the complete solution of the minimalization problem for each given set of values of the limitational factors $u_p, u_q \cdots u_t$, so that at least this part of the work need not be done by a series of tentative steps.

We shall in particular consider the case where only rationing coefficients $x_1 \cdots x_n$ are introduced (besides the partakers' percentages $z_1 \cdots z_n$). In this case it seems to be a reasonable working hypothesis to assume that Ω varies as indicated by the curves in Fig. 1 of Section 7. This means that we put

$$2\Omega = \sum_{k=1,2\cdots n} \frac{x_k^2}{\epsilon_k} \cdot \tag{8.1}$$

In this case, the increment in total inconvenience is defined, not only for infinitesimal steps, but for finite movements. In the present adaptation process, we need not, therefore, confine the discussion to a series of small displacements in x_k space, but we may study what can be obtained by going directly to some *a priori* given point in this space. Also for the limitational factors $u_1 \cdots u_n$ we may now, if we wish, consider large jumps, but in practice this will not always be found convenient, at least in parts of the adaptation process it will be found useful to work with small tentative variations of the $u_1 \cdots u_n$. The calculations will take the following form.

First, let a certain magnitude C of the total volume of transactions be given. By the principles of the preceding Sections we associate with this magnitude C a certain set of conditional parameters $u_p, u_q \cdots u_t$ whose values are determined when C is given. The problem will then be to minimize (8.1) under the side conditions

$$\sum_{k=1,2\cdots n} f_{ik}x_k = u_i \qquad (i = p, q \cdots t) \tag{8.2}$$

where the u_i are given.

If we let $\lambda_p, \lambda_q \cdots \lambda_t$ be a set of m arbitrary parameters and consider the derivatives

$$\frac{\partial\left(\Omega - \sum_{i=p,q\cdots t} \lambda_i \sum_{h=1,2\cdots n} f_{ih}x_h\right)}{\partial x_k} = \frac{x_k}{\epsilon_k} - \sum_{i=p,q\cdots t} \lambda_i f_{ik} \tag{8.3}$$

we see that the necessary condition for a minimum is

$$x_k = \sum_{i=p,q\cdots t} \lambda_i f_{ik} \qquad (k = 1, 2 \cdots n) \tag{8.4}$$

where

$$\bar{f}_{ik} = f_{ik}\epsilon_k. \tag{8.5}$$

The values of the λ_i that correspond to the extremum point are determined by inserting into (8.2) the expression for x_k taken from (8.4). This gives

$$\sum_{j=p,q\cdots t} \phi_{ij}\lambda_j = u_i \qquad (i = p, q \cdots t) \tag{8.6}$$

where

$$\phi_{ij} = \sum_{k=1,2\cdots n} f_{ik}\bar{f}_{jk} = \sum_{k=1,2\cdots n} \epsilon_k f_{ik} f_{jk}. \tag{8.7}$$

So long as the number m of affixes in the set $p, q \cdots t$ is less than n, the m-rowed and n-columned matrix f_{ik} will not be singular (except by coincidence). Since we do not consider the case where *all* the $u_1 \cdots u_n$ are included in the conditional set, we may consequently assume that f_{ik} is of rank m, and hence the linear system (8.6) non-singular

Let

$$\hat{\phi}_{ij(p,q\cdots t)} = \text{the adjoint of the } m\text{-rowed matrix } \phi_{ij}, \tag{8.8}$$

the adjunction being made within the set $p, q \cdots t$. Further let

$$|\phi_{(p,q\cdots t)}| = \text{determinant value of the coefficients in the left member of (8.6).} \tag{8.9}$$

With this notation we have

$$\lambda_i = \sum_{j=p,q\cdots t} \hat{\phi}_{ij(p,q\cdots t)}u_j/|\phi_{(p,q\cdots t)}| \qquad (i = p, q \cdots t). \tag{8.10}$$

When $\lambda_p, \lambda_q \cdots \lambda_t$ are computed by this formula, all the $x_1 \cdots x_n$ are determined by (8.4), hence by (7.7) all the $u_1 \cdots u_n$ are determined (not only those corresponding to $i = p, q \cdots t$). It is easily seen that these values of $u_1 \cdots u_n$ are obtained simply by using (8.6) also for all the other values of i in the set $1, 2 \cdots n$. The definition (8.7) applies, of course, for any values of $i, j = 1, 2 \cdots n$.

This computation of the complete set of u's tells us whether a deficiency factor has come in by the minimalization process. If no such factor appears, the selection of the conditional set $u_p, u_q \cdots u_t$ has been successful, the point x_k thus determined will then actually represent a minimum of Ω under the given magnitude of C.

If a deficiency factor has come in, the computation will in any case yield information about a situation in x_k space that is not very far from a minimum combination, but it will not give exactly such a point;

if a more exact result is wanted, another selection of the conditional set $u_p, u_q \cdot \cdot \cdot u_t$ must therefore be tried, possibly in connection with a small modification in the magnitude of C for which the minimalization is carried through.

The value which Ω assumes in the point x_k that is fixed by the above procedure, may be easily expressed in terms of the corresponding values of $\lambda_p, \lambda_q \cdot \cdot \cdot \lambda_t$. Inserting into (8.1) the expression for x_k taken from (8.4) we get

$$2\Omega = \sum_k \frac{1}{\epsilon_k}\left(\sum_i \lambda_i \bar{f}_{ik}\right)^2 = \sum_k \sum_{ij} \frac{1}{\epsilon_k}\lambda_i\lambda_j\bar{f}_{ik}\bar{f}_{jk}$$

where k runs through $1, 2 \cdot \cdot \cdot n$, and i, j, independently of each other, through $p, q \cdot \cdot \cdot t$; hence

$$2\Omega = \sum_{ij} \phi_{ij}\lambda_i\lambda_j \tag{8.11}$$

where i, j runs through $p, q \cdot \cdot \cdot t$. By (8.6) the expression (8.11) can also be written

$$2\Omega = \sum_{i=p,q\cdots t} \lambda_i u_i. \tag{8.12}$$

The last formula is very convenient for computation purposes, since both the λ_i and u_i are already known when we get to the stage where it is desired to compute Ω.

The above computations may be made for a series of different values of C. To each chosen magnitude of C there will correspond a value of Ω. The curve exhibiting this relation between Ω and C is the optimum curve. *It shows the inevitable total inconvenience of adjustment which must be incurred if a given total volume of transaction for the community shall be assured.*

Under the assumption here made, the optimum curve is continuous, and it has even a continuously varying tangent, except at the points where one or more new minimum factor comes in. It follows from the above argument that as C increases, a larger and larger number of factors $u_1 \cdot \cdot \cdot u_n$ must be included in the conditional set; for each value C where a new factor is thus included, the tangent of the optimum curve changes discontinuously.

Under our assumptions the optimum curve will everywhere be sloping upwards, that is to say Ω will be constantly increasing with C, and the discontinuous changes in the slope of the tangent will always be an *increase* in the steepness. In most practical cases the element of discontinuity introduced by the inclusion of new minimum factors will not be very prominent; it will indeed hardly be perceptible unless the com-

putations are carried through with a great number of decimal places and the graph is drawn on a very large scale; Figure 1 in Section 9 is an example of this.

The feature which will dominate the picture is the *rapid increase in* Ω *that takes place after a certain level of* C *is reached.* This is explained by the fact that for higher C a rapidly increasing number of minimum factors must be taken account of, and this imposes *new side conditions* that become more and more severe, thus making it more and more difficult to bring Ω down to a reasonable magnitude. This is clearly shown in the numerical example of the next Section.

9. *A Numerical Example Showing Details of the Computation*

In order to illustrate the use of the technique developed in the preceding sections we shall now work through in detail a simplified numerical example.

Suppose that there is given the three-rowed request-matrix (4.15). If only partakers' percentages z_1, z_2, z_3 are used, in other words, if it is required that the *relative* distribution of each person's purchase shall be maintained unaltered, then total sales will, as we have seen, only be 975 and the individual sales will be given by (4.21). The problem before us is how we can, by modifying somewhat the relative distribution for each partaker, under due consideration of his desires, increase total sales. For simplicity we shall assume that only rationing coefficients x_1, x_2, x_3 are employed, (besides the partakers' percentages z_1, z_2, z_3) and that the total inconvenience.in applying such a set of coefficients may be represented by a function of the form (8.1). As an illustration we put

$$\begin{aligned} \epsilon_1 &= 0.1 \\ \epsilon_2 &= 0.2 \\ \epsilon_3 &= 0.7. \end{aligned} \tag{9.1}$$

The first thing to do is to form the row-relative request-matrix, namely

(9.2)

α_{ij}	$j=1$	2	3	Row check:
$i=1$	.00000	.33333	.66667	1.00000
2	.18182	.00000	.81818	1.00000
3	.39286	.60714	.00000	1.00000

Similarly we form the column-relative request matrix, namely

(9.3)

β_{ij}	$j=1$	2	3
i=1	.00000	.12821	.29412
2	.19512	.00000	.70588
3	.80488	.87180	.00000
Column check:	1.00000	1.00000	1.00000

On the basis of these two matrices we form $\gamma_{ij}=\sum_k \alpha_{ki}\,\beta_{kj}$. The result is given in (9.4)

(9.4)

γ_{ij}	$j=1$	2	3
$i=1$	.35168	.34250	.12834
2	.48867	.57204	.09804
3	.15964	.08547	.77362
Column check by formula (6.9)	.99999	1.00001	1.00000

The small deviations from unity in the column sums in (9.4) are due to the dropping of decimal places.

Subtracting e_{ij} from (9.4) we get

(9.5)

$\gamma_{ij}-e_{ij}$	$j=1$	2	3
$i=1$	−.64832	.34250	.12834
2	.48867	−.42796	.09804
3	.15964	.08547	−.22638
Column check by formula (6.9)	−.00001	.00001	.00000

The above fundamental matrices are derived by straightforward product-summing, a work that may be easily performed even in the case of a large number of variables.

Starting from (9.2) we now form the matrix $A_{ij}=A_{ij}^{(1)}=\alpha_{ij}+e_{ij}/2$ and its successive powers $A_{ij}^{(\nu)}$ defined by (22.9). In all these matrices each row shall add up to unity. If a listing adding machine is used and there is only a small number of variables, it is quicker to verify that the sum of the three column-sums is equal to the number of variables, in the example considered. This is done in Table 1.

TABLE 1.—ITERATION PROCESS $A_{ij}^{(\nu)}$

	.50000	.16667	.33333
$\nu=1$	.09091	.50000	.40909
	.19643	.30357	.50000
3.00000	.78734	.97024	1.24242
	.33063	.26786	.40151
$\nu=2$	.17127	.38934	.43939
	.22403	.33631	.43966
3.00000	.72593	.99351	1.28056
	.26854	.31092	.42054
$\nu=3$	.20734	.35660	.43606
	.22895	.33896	.43209
3.00000	.70483	1.00648	1.28869
	.24514	.32788	.42698
$\nu=4$	.22175	.34523	.43302
	.23017	.33881	.43102
3.00000	.69706	1.01192	1.29102
	.23625	.33442	.42933
$\nu=5$	.22732	.34103	.43165
	.23055	.33861	.43084
3.00000	.69412	1.01406	1.29182
	.23286	.33692	.43022
$\nu=6$	.22945	.33944	.43111
	.23069	.33852	.43079
3.00000	.69300	1.01488	1.29212
	.23157	.33787	.43056
$\nu=7$	.23027	.33883	.43090
	.23074	.33848	.43078
3.00000	.69258	1.01518	1.29224
	.23108	.33824	.43068
$\nu=8$	.23058	.33860	.43082
	.23076	.33847	.43077
3.00000	.69242	1.01531	1.29227
	.23089	.33838	.43074
$\nu=9$	.23070	.33852	.43077
	.23077	.33846	.43077
3.00000	.69236	1.01536	1.29228

	.23082	.33843	.43076
$\nu=10$	.23074	.33848	.43077
	.23077	.33846	.43077
3.00000	.69233	1.01537	1.29230
	.23078	.33845	.43077
$\nu=11$	.23076	.33847	.43077
	.23077	.33846	.43077
3.00000	.69231	1.01538	1.29231
	.23077	.33846	.43077
$\nu=12$	.23077	.33846	.43077
	.23077	.33846	.43077
3.00000	.69231	1.01538	1.29231

Table 1 exhibits clearly how the iteration process converges. The elements of the final matrix $A_{ij}^{(\infty)}$ obtained must by (22.14) be proportional to those of the adjoint of $(\alpha_{ij}-e_{ij})$. The rows of this adjoint are not only proportional (which follows simply from the fact that $(\alpha_{ij}-e_{ij})$ is singular), but they are (by Lemma II of Section 21) even *equal*. This is clearly seen by considering the last part ($\nu=12$) in Table 1. Furthermore, the elements in one of these equal rows are just the numbers (4.18) that were determined directly as the proportionality numbers for the adjoint of $(\alpha_{ij}-e_{ij})$ in the present example.

In the present simple case these proportionality numbers are easily determined either directly as in (4.17), or by the iteration process in the above Table 1; the direct method is here the simplest, but in the case of a great number of variables it is the iteration process that must be relied upon.

The next step is to construct the matrix Q_{ij}. For this purpose we first form the differences $D_{ij}^{(\nu)}$ defined by (22.18); they are given in Table 2. The check in this case is similar to the one used in Table 1, only the sum in each row should now be zero, and hence the sum of the column-sums should also be zero; it will be seen that this checks everywhere in the table.

TABLE 2.—SERIES $D_{ij}^{(\nu)}$.

	−.50000	.16667	.33333
$\nu=1$	.09091	−.50000	.40909
	.19643	.30357	−.50000
.00000	−.21266	−.02976	.24242
	−.16937	.10119	.06818
$\nu=2$	.08036	−.11066	.03030
	.02760	.03274	−.06034
.00000	−.06141	.02327	.03814

	−.06209	.04306	.01903
$\nu=3$	.03607	−.03274	−.00333
	.00492	.00265	−.00757
.00000	−.02110	.01297	.00813
	−.02340	.01696	.00644
$\nu=4$	.01441	−.01137	−.00304
	.00122	−.00015	−.00107
.00000	−.00777	.00544	.00233
	−.00889	.00654	.00235
$\nu=5$	.00557	−.00420	−.00137
	.00038	−.00020	−.00018
.00000	−.00294	.00214	.00080
	−.00339	.00250	.00089
$\nu=6$	.00213	−.00159	−.00054
	.00014	−.00009	−.00005
.00000	−.00112	.00082	.00030
	−.00129	.00095	.00034
$\nu=7$	.00082	−.00061	−.00021
	.00005	−.00004	−.00001
.00000	−.00042	.00030	.00012
	−.00068	.00051	.00017
$\nu=8$	.00043	−.00031	−.00012
	.00003	−.00002	−.00001
.00000	−.00022	.00018	.00004
	−.00019	.00014	.00006
$\nu=9$	.00012	−.00008	−.00005
	.00001	−.00001	.00000
.00000	−.00006	.00005	.00001
	−.00007	.00005	.00002
$\nu=10$	.00004	−.00004	.00000
	.00000	.00000	.00000
.00000	−.00003	.00001	.00002
	−.00004	.00002	.00001
$\nu=11$	.00002	−.00001	.00000
	.00000	.00000	.00000
.00000	−.00002	.00001	.00001
	−.00001	.00001	.00000
$\nu=12$	.00001	−.00001	.00000
	.00000	.00000	.00000
.00000	.00000	.00000	.00000

$D_{ij}^{(13)} = .00000$

From the data in Table 2 the successive approximations to a given element in the matrix Q_{ij} are formed by continuous summation, using (22.17). When a listing adding machine is used the result will come in the form of a column of figures (the successive approximations) for each given (i, j) combination. It is therefore now more convenient to tabulate the result in the form used in Table 3.

TABLE 3.—SUCCESSIVE APPROXIMATIONS TO Q_{ij}.

	$j=-1$	2	3
	−.50000	.16667	.33333
	−.83874	.36905	.46969
	−1.02501	.49823	.52678
	−1.11861	.56607	.55254
	−1.16306	.59877	.56429
	−1.18340	.61377	.56963
$i=1$	−1.19243	.62042	.57201
	−1.19787	.62450	.57337
	−1.19958	.62576	.57391
	−1.20028	.62626	.57411
	−1.20072	.62648	.57422
	−1.20084	.62660	.57422
	.09091	−.50000	.40909
	.25163	−.72132	.46969
	.35984	−.81954	.45970
	.41748	−.86502	.44754
	.44533	−.88602	.44069
2	.45811	−.89556	.43745
	.46385	−.89983	.43598
	.46729	−.90231	.43502
	.46837	−.90303	.43457
	.46877	−.90343	.43457
	.46899	−.90354	.43457
	.46911	−.90366	.43457
	.19643	.30357	−.50000
	.25163	.36905	−.62068
	.26639	.37700	−.64339
	.27127	.37640	−.64767
	.27317	.37540	−.64857
3	.27401	.37486	−.64887
	.27436	.37458	−.64894
	.27460	.37442	−.64902
	.27469	.37433	−.64902
	.27469	.37433	−.64902
	.27469	.37433	−.64902
	.27469	.37433	−.64902

Each of the 9 cells in Table 3 represents one of the elements of Q_{ij}, the first figure in the cell being $(A_{ij}-e_{ij})$, and the last the 12th approximation to Q_{ij}. On each level of approximation we have the same

check as before, namely that the sum of the elements in a given row is zero.

The next step is to compute the matrix whose (i, j)th element is $\sum_k(\gamma_{kj}-e_{kj})Q_{ki}$. It is convenient to write this matrix in such a way that the second affix on Q designates the row number and the second affix on γ the column number. The result is given in (9.6).

(9.6)

$\sum_k(\gamma_{kj}-e_{kj})Q_{ki}$	$j=1$	2	3
$i=1$	1.05162	−.58857	−.17031
2	−.78807	.63333	−.09292
3	−.26353	−.04478	.26323
Column check by (22.19)	.00002	−.00002	.00000

From the matrix (9.6) we compute easily f_{ij} by (7.3). The result is given in (9.7). Each element in (9.7) is obtained from the corresponding element in (9.6) by multiplication with the row and column factors indicated by (7.3).

(9.7)

f_{ij}	$j=1$	2	3
$i=1$	−934.18623	746.01018	188.19197
2	477.32202	−547.32936	70.00712
3	125.41158	30.40631	−155.82218
Column check by (7.5)	−.00420	.00580	.00015

From (9.7) we get $\bar{f}_{ij}$ by (8.5), the result is given in (9.8).

(9.8)

$\bar{f}_{ij}$	$j=1$	2	3
$i=1$	−93.41862	149.20204	131.73438
2	47.73220	−109.46587	49.00498
3	12.54116	6.08126	−109.07553
Column check: $\bar{f}_{0j}=\epsilon_j f_{0j}$	−33.14526	45.81743	71.66383

Finally the symmetric matrix ϕ_{ij} is computed by (8.7), the result is given in (9.9)

(9.9)

ϕ_{ij}	$j=1$	2	3
$i=1$	223368	−117031	−27706
2	−117031	86128	−4978
3	−27706	−4978	18754
Column check: $\sum_i P_i\phi_{ij}=0$	1.40	−.73	.09

In the three subsets (12), (13) and (23) the adjoints of ϕ_{ij} are

(9.10)

$\hat{\phi}_{ij(12)}$ (symmetric)	$j=1$	2
$i=1$	86128	
2	117031	223368

(9.11)

$\hat{\phi}_{ij(13)}$ (symmetric)	$j=1$	3
$i=1$	18754	
3	27706	223368

(9.12)

$\hat{\phi}_{ij(23)}$ (symmetric)	$j=2$	3
$i=2$	18754	
3	4978	86128

The determinant values are

$$|\phi_{(12)}| = 5542.10^6 \tag{9.13}$$

$$|\phi_{(13)}| = 3421.10^6 \tag{9.14}$$

$$|\phi_{(23)}| = 1590.10^6. \tag{9.15}$$

The check on the computation of the adjoint (9.10) is that the same value (9.13) is obtained by taking the product sum of the first row of (9.10) with corresponding elements of (9.9) as by taking the similar product sum in the second row. The adjoints (9.11) and (9.12) are checked in a similar way.

We now have the necessary tools to construct the optimum curve. The calculations are best arranged in a scheme as indicated in Table 4.

TABLE 4.—CONSTRUCTION OF THE OPTIMUM CURVE

	Starting point	1. Computed Point	2. Computed Point
u_1	0.0000	50.0000	100.0000
u_2	0.0000	−26.1969	−52.3938
u_3	0.0000	−6.2019	−12.4037
λ_1	0.0000	.0002238459	.0004476917
λ_2	0.0000		
λ_3	0.0000		
Check by (9.16)	0.0000	0.0000	0.0000
Check $\sum_i u_i P_i = 0$	0.0000	0.0000	0.0000
C_1	975.0000	1025.0000	1075.0000
C_2	1300.0000	1273.8031	1247.6062
C_3	1950.0000	1943.7981	1937.5963
Check $\sum_i C_i P_i - a_{00}0$	0.0000	0.0006	0.0009
Abscissa $C = \min [C_i]$	975.0000	1025.0000	1075.0000
Ordinate $2\Omega = \sum_i u_i \lambda_i$	0.0000	0.0112	0.0448

As the starting point on the optimum curve we take the one corresponding to $\Omega = 0$. In this point all the limitational factors u are zero; the quantities C_i are therefore by (6.20) simply the numbers (4.19), and C, the smallest of the numbers C_i, consequently 975. The situation in the starting point is thus nothing but the situation obtained when only partakers' percentages are used. The corresponding figures are inserted in the first column of Table 4.

From the magnitudes C_i in the first column of 4 is seen that it is here No. 1 that is the minimum factor; consequently if a move along the optimum curve is to be made, we must start by making u_1 a positive quantity.

If we did not need to take account of the fact that C_2 and C_3 will be lowered by the minimalization process (i.e., that u_2 and u_3 will become negative) we could immediately put u_1 so large as to bring the factor No. 1 on the same level as the next factor, which means that we would put $u_1 = 325$. But such a choice would certainly introduce at least No. 2 as a deficiency factor in the new point computed, thus keeping the total volume below the level 1300 which corresponds to the choice $u_1 = 325$. We shall therefore move by shorter steps, say, first increasing C_1 by 50, i.e., putting $u_1 = 50$. Moving by so small a step has also the advantage that we get information about the shape of the optimum curve in more frequent points.

The computations for the new point are carried out thus: first the figure $u_1 = 50$ is entered on the first line in the column headed "1. computed point" in Table 4. We now consider u_1 as the only conditional factor, and determine the corresponding λ_1 by (8.10) which in the present case simply means putting $\lambda_1 = u_1/\phi_{11} = 0.0002238459$. This figure is entered on the corresponding line in the column in question. An application of the sum check

$$(9.16) \qquad \sum_{j=p,q\cdots t} \phi_{0j(p,q\cdots t)}\lambda_j - (u_p + u_q + \cdots + u_t) = 0$$

where

$$(9.17) \qquad \phi_{0j(p,q\cdots t)} = \sum_{i=p,q\cdots t} \phi_{ij}$$

may, for the sake of principle, be used even in this simple case where there is only one conditional factor.

The parameter λ_1 being determined, the two remaining factors u_2 and u_3 are computed (by applying (8.6) also for $i = 2, 3$), and inserted in their place, whereafter the check $\sum_i u_i P_i = 0$, i.e., $u_1P_1 + u_2P_2 + u_3P_3 = 0$ is applied and entered in its proper place. Adding the quantities u_1, u_2, u_3, to a_{10}/P_1, a_{20}/P_2, a_{30}/P_3 respectively we get the new quantities $C_1 = 1025$, $C_2 = 1273.8031$, and $C_3 = 1943.7981$ on which the check sum $\sum_i C_i P_i - a_{00} = 0$ is run.

It thus turns out that the smallest of the C_i is *still* No. 1, which means that no deficiency factor has come in by the movement from the starting point to the first computed point. The first computed point is therefore the true minimum point which will be obtained when it is required that the total volume C shall be kept constant at 1025. The magnitude $C = 1025$ is the abscissa of the point considered on the optimum curve. Its ordinate (affected with the conventional factor 2) is obtained by taking the product sum $\sum_i u_i \lambda_i$ which gives 0.0112.

It is seen that the reaction in C_2 and C_3 corresponding to the chosen increase of 50 in the minimum factor C_1 is only small; indeed, C_2 has only dropped from 1300 to about 1274 and C_3 only from 1950 to about 1944; there is thus still a long way to go until any of the factors Nos. 2 or 3 become minimum factors. If only a rapid orientation is needed, it therefore now appears that there would be no risk in taking a bigger step during which factor No. 1 is still kept as the only conditional factor. We may put for instance $u_1 = 175$, which means putting $C_1 = 1150$. We shall, however, continue by small steps in order to determine a tighter set of points on the optimum curve. Putting in the next point $u_1 = 100$, that is $C_1 = 1075$, we can repeat the same computations as in the previous column. The result is that we determine a new point where $C = 1075$, and $2\Omega = 0.0448$. This point has also, as we expected,

turned out to be a true minimum point, because the factor that was kept conditional (No. 1) remains as minimum factor in the point reached by the computation.

In this way we can continue. It turns out that when u_1 is equal to about 213 the second factor comes very closely down to being equal to the first, in other words, we are now very near a point where there are two minimum factors. The situation is indicated by the first column of Table 5.

TABLE 5.—CONSTRUCTION OF THE OPTIMUM CURVE CONTINUED

	6. point	7. point	8. point
u_1	213.2000	213.2632	313.2632
u_2	−111.7036	−111.7367	−11.7367
u_3	−26.4448	−26.4526	−158.5931
λ_1	0.0009544787	0.0009547616	0.0046205657
λ_2			0.0061421693
λ_3			
Check by (9.16)	0.0000002	0.00001	0.0009
Check $\sum_i u_i P_i = 0$	0.0013	0.0014	0.0022
C_1	1188.2000	1188.2632	1288.2633
C_2	1188.2964	1188.2633	1288.2633
C_3	1923.5552	1923.5474	1791.4069
Check $\sum_i C_i P_i = a_{00}$	0.0016	0.0016	0.0025
Abscissa $C = \min [C_i]$	1188.2	1188.2632	1288.2633
Ordinate $2\Omega = \sum_i u_i \lambda_i$	.2035	.2036	1.3754

Making the very small step from point 6 to point 7 in table 5—still with only No. 1 as conditional factor—we reach a point where the two factors 1 and 2 coincide exactly in being minimum factors (within the accuracy carried in the computation). The point 7 where $C = 1188.2632$ may thus be taken as the exact point where factor No. 2 comes in.

Now let us decide to make a larger step forward, keeping this time *both No. 1 and No. 2 as conditional factors*. Let us, for instance, increase both these minimum factors by 100 as compared with the values they had in point 7. This means that we put $u_1 = 313.2633$ and $u_2 = -11.7367$. These figures are entered on the two first lines in the column for the 8. point. We have now to reckon with two λ parameters, namely λ_1 and λ_2. They are determined by (8.10), and from these values of the λ's are the other magnitudes in the 8. point determined. The result is given in the last column of Table 5.

It is seen that the factor that was now considered unconditional, namely the third factor, is still very far from being minimal, point 8 is therefore a true point on the optimum curve.

Continuing in this way we get a series of points which give the shape of the optimum curve exhibited in Figure 1. In all 15 points were computed on the curve.

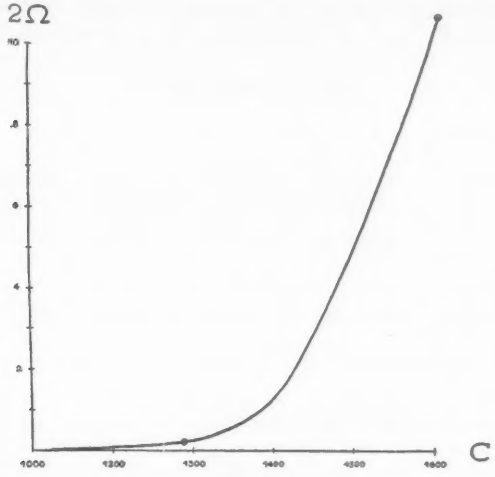


FIGURE 1

From the shape of the curve in Figure 1 is seen that the total inconvenience increases rapidly after C has passed 1200. It therefore seems indicated to fix the final adaptation in the vicinity of this point. Let us, for instance, choose the point 7 in table 5, where $C = 1188.2632$ (this happened to be just the point where factor No. 2 becomes minimal).

In the point thus selected as the final optimum point we first have to compute the rationing percentages x_1, x_2, x_3 which in the present case reduce to $x_i = \bar{f}_{i_1}\lambda_1$. The result is

(9.18)
$$\begin{aligned} x_1 &= -0.089193 \\ x_2 &= 0.142452 \\ x_3 &= 0.125775 \\ \text{Total}\quad x_0 &= 0.179035. \end{aligned}$$

As a check on the computation we have

(9.19)
$$\sum_{j=p,q\cdots t} f_{0j}\lambda_j = x_0$$

where, as before, the affix zero denotes the result obtained by performing a summation to the affix in question.

The values in (9.18) tell us that in the final adaptation point there is an *actual rationing* of the shoemaker's product (No. 1) while there is distributed an extra quota of the products of the tailor (No. 2) and the farmer (No. 3).

The rationing percentages being computed, we form the *rationing matrix* $a_{ij}\,x_j$; the result of this computation is given in (9.20).

THE RATIONING MATRIX

(9.20)

$a_{ij}x_j$	$j=1$	2	3	v_i
$i=1$	0.00000	10.68393	18.86624	29.55017
2	− 7.13540	0.00000	45.27897	38.14357
3	−29.43353	72.65071	0.00000	43.21718
Check: $a_{0j}x_j$	−36.56893	83.33464	64.14521	110.91092($=v_0$)

The column sums in (9.20) ought to be equal to $a_{0j}x_j$.

The row sums in (9.20) we denote

$$v_i = \sum_j a_{ij}x_j. \tag{9.21}$$

The computation of the sums (9.21) is checked by verifying that the sum of row sums and the sum of column sums in (9.20) are equal.

On the basis of C and the u_i as determined in the optimum point (point 7 in Table 5) and the values (4.18) we compute the total purchases c_i of the partakers by the formula (6.17). The result is

(9.22)

$$\begin{aligned} c_1 &= 225.00 \\ c_2 &= 440.00 \\ c_3 &= 523.26 \\ \text{Total } c_0 = C &= 1188.26. \end{aligned}$$

As a check on this computation we verify that the sum of the c_i is equal to the value of C in the optimum point. It is seen that the sum checks for the decimal places used.

Further it is seen from (9.22) that both the partakers No. 1 and No. 2 get a total purchase *exactly equal to the one they originally asked for* (compare (4.15)). This is only another expression for the fact that *we have now gone along the optimum curve far enough to reach a point where*

both Nos. 1 and 2 become minimum factors (which means that both u_1 and u_2 must be considered as conditional under a *further* outward movement on the optimum curve).

By means of the values (9.22) and the row sums (9.21) it is now easy to compute the partakers' percentages

$$1 + z_i = \frac{c_i - v_i}{a_{i0}}. \tag{9.23}$$

The result is given in (9.24)

$$\begin{aligned} 1 + z_1 &= .868662 \\ 1 + z_2 &= .913305 \\ 1 + z_3 &= .571483. \end{aligned} \tag{9.24}$$

As a check on the computation of (9.24) we have

$$\sum_i (1 + z_i)a_{i0} - C + v_0 = 0 \tag{9.25}$$

where $v_0 = \sum_i v_i$ is the grand total of the elements in (9.20).

Carrying out the check (9.25) we get 0.00135 which is zero within the limits of accuracy used in the computations. Indeed, the order of magnitude of the terms involved in (9.25) is between 200 and 500.

The fact that both $(1+z_1)$ and $(1+z_2)$ fall below unity must, of course, not be interpreted as indicating that neither partaker No. 1 nor No. 2 utilize their full capacity, indeed, in the present case where also rationing coefficients are used, the percentages $(1+z_i)$ do not—as in the simple case considered in Section 4—indicate the capacities used. As we have seen both Nos. 1 and 2 are actually in the present case used to full capacity.

By means of the partakers' percentages $(1+z_i)$ we compute the partakers' matrix $(1+z_i)\,a_{ij}$, which is given in (9.26).

THE PARTAKERS' MATRIX

(9.26)

$(1+z_i)a_{ij}$	$j=1$	2	3	Check: Sum of row no. i equal to $(1+z_i)a_{i0}$
$i=1$	0.00000	65.15019	130.30039	195.45058
2	73.06444	0.00000	328.78996	401.85440
3	188.58948	291.45646	0.00000	408.04294

Finally by taking the sum of the matrices (9.20) and (9.26) we get the completely corrected matrix (9.27).

THE COMPLETELY CORRECTED MATRIX

	$(1+z_i)a_{ij}+a_{ij}x_j$	Is allowed to buy from			Total
		Shoemaker	Tailor	Farmer	
(9.27)	Shoemaker	0.00	75.83	149.17	225.00
	Tailor	65.93	0.00	347.07	440.00
	Farmer	159.16	364.11	0.00	523.26
	Total	225.08	439.94	523.24	1188.26

The check on (9.27) consists in verifying that the sum of the first row is equal to the sum of the first column, and similarly for the second row and column and for the third row and column. It is seen that this checks within the accuracy carried. Finally, it should be checked that the grand total of (9.27) is equal to the abscissa C of the optimum curve in the point that was selected as the equilibrium point (point 7 in Table 5).

10. Estimate of the Amount of Work Involved when the Number of Variables is Very Great

The computations of the preceding sections can be easily carried through when the number of commodities (or services) involved is small. What will be the situation if this number is very large? Will the amount of work be prohibitive? I believe it will not. Let us try to make a rough estimate of the work.

The bulk of the work involved will consist of taking large product sums. For instance, nearly all the preceding formulae expressing one variable in terms of others are *linear forms*. And the inversion processes for the matrices considered are also built on a repeated use of product sums; this is, for instance, the case with the series leading to the determination of the matrix Q_{ij}. In practice some similar process, for instance the Neumann-series, will have to be used in order to determine the adjoint matrices of the ϕ's in the construction of the optimum curve. If only a rough estimate is desired, it therefore seems plausible to measure the amount of work by the number of multiplications involved.

The number of multiplications will for some of the computations be proportional to the number of variables n, for others it will be proportional to n^2 and for still others proportional to n^3. If we follow the

method of Section 9 the handling of the basic matrices require the following number of multiplications:

(10.1)

n^3	n^2
	matrix α
	matrix β
matrix γ	
each step in the series for matrix Q	
matrix f	
	matrix $\bar{f}$
matrix ϕ	

If n is a large number, say 50, we need not take account of those parts of the work where the number of multiplications is proportional either to n or n^2; we get a sufficiently accurate estimate by considering only those phases of the work that are listed in the first column of (10.1). If ν steps are required in the iteration process performed to get matrix Q, the total number of multiplications involved in the construction of the basic matrices will therefore be

$$(\nu + 3)n^3. \tag{10.2}$$

For the determination of one point on the optimum curve we need a Neumann-series, or a similar series, by which to invert the m-rowed submatrix ϕ, m being the number of conditional factors. This involves a number of multiplications equal to m^3; if μ steps are needed before the process converges, this gives μm^3 multiplications. We may therefore conclude that the total number of multiplications needed to construct the optimum curve by means of the basic matrices is

$$\omega\mu m^3 \tag{10.3}$$

where ω is the number of points on the curve, m the number of conditional factors needed in the determination of one such point and μ the average number of steps needed in the convergency process.

The computations necessary to finish the work and finally to arrive at the corrected request-matrix, only involve a number of multiplications proportional to n^2 and may therefore here be neglected.

As an example, suppose that $n = 50$, and that $\nu = 7$ steps are needed in the iteration process performed on Q. This gives by (10.2) a total number of multiplications equal to 1250000. From the time records in my statistical laboratory at the University Institute of Economics in Oslo, it appears that one should strike a safe average by saying that

when everything is included: preparatory steps (arrangement of tables, etc.), checkings, search for errors, occasional rests, etc., 100 products of 6 digit numbers take on the average 1 hour. Of course, for any particular small job, the time used may be considerably less, but a safe long time average would, I think, be somewhere near the one mentioned. On the basis of this the handling of the basic matrix would require about 300 weeks' work.

The construction of the optimum curve would in all probability take somewhat less; indeed, the number of variables which on the average is included in the conditional set would as a rule hardly be larger than one-third or one-half of n; and it would seldom be necessary to determine more than, say, 6 or 7 points on the curve. At any rate it seem that this item, as well as the other smaller items would be largely covered by doubling the 300 weeks previously found, thus giving 600 weeks. If the work is to be completed in 2 weeks time, a staff of some 300 computors would thus be needed.

I have here not made any allowance for the use of special labor saving devices that may be developed. If the work was organized on a permanent basis, it is quite obvious that such devices would be worked out to great perfection. One only has to think of the possibility of punching two factors separately on two loose halves of Hollerith cards; in the present case where repeated product sums are used *with one fixed factor*, this would be a great help. Various kinds of special short cut arrangements of the computations would also undoubtedly be invented as experience was gained in the actual work.

But even assuming that a staff of 300 people is needed, would this be a big item to reckon with? I think it is right to say that it would—even in a small country—be nothing but a *quantité négligeable* if the system actually was able to secure some of the results aimed at.

PART II. APPLICATION

11. *Introduction: General Remarks on the Practical Organization of the Planned Exchange Service*

The administration of a planned exchange service built on the principles expounded in the preceding sections will of course raise a series of practical problems. Will it be possible to solve them in such a way that the system actually becomes practicable? I believe it will.

It is well known that in various parts of the world (particularly in the United States and Germany) isolated attempts have been made to establish exchange organizations or introduce new types of money-substitutes which have had the object of pressing forward that exchange of goods and services which under the old system had stagnated. According to the reports which are available these measures

seem to have done much good. They have, however, all been of a more or less local and sporadic character. None of them have been organized on a national basis. There is reason to believe that if such measures were organized on a national basis, and supported by federal or state authority, they would be capable of yielding great results.

If the economic activity *within* each of the big economic groups in the country, for instance, within the shoe industry, the textile industry, agriculture, etc., had been perfectly planned and controlled, the problem of the national exchange services would only have been to organize the circulation *between* the groups. This could have been done almost exactly according to the method of the preceding sections.

The activity within each such group is not, however, at present organized as a planned unity. The problem is thus how the *individual* persons or concerns within a group can be fitted into the exchange service.

It is clear that it will not be possible to use an accounting system which is so complete that each individual person, who can be expected to become a participant in the exchange transaction, receives an individual account, which is balanced against all the other accounts by the preceding methods.

It is only the *large groups* which can be treated in this manner. But at the same time something must be done so that there is established an equity between services and counterservices also for the individual persons or concerns within a group. It is the arrangement of this question which is the actual practical problem in the carrying out of the exchange system. The principle according to which a solution must be sought, must, in my opinion, be the following: The warrants are drawn on one of the large groups, not on a particular person. The distribution, of the warrants, on the other hand, must be made to particular persons, perhaps, through the medium of certain centrals (see below). A complete account must be kept of what the individual persons or concerns are allocated of goods warrants. This being done, efforts must be made to arrange matters in such a way that the warrants *by themselves*, more or less automatically flow through the usual trade channels and finally find their way to the proper suppliers who participate in the exchange service. The following is a suggestion of how this may be carried out in practice.

12. Membership in the National Exchange Organization

To a certain extent the technique in the organization of such an exchange service will resemble that which was employed in the rationing during the war. There is, however, the difference that the exchange service should be based on voluntary participation, and must embrace

many more kinds of goods and services. It will first obtain its full effectiveness when there is a comparatively large choice of goods and services.

Everybody who desires to participate in the exchange service, individual persons or concerns, ought to subscribe as *members* of a particular branch (textile, agriculture, etc.). The nomenclature of this division is an important question which would have to be looked into with great care. Transversally of this division there must be a geographical division. The most expedient would presumably be that, in each city or country district, there be established an exchange office. To commence with, this could assume a very modest form. It may perhaps be arranged simply by some taxation officer undertaking to keep the necessary lists. If a member had a considerable interest in more than one group of suppliers, it would probably be more practical to let him enter with a separate membership into each of the groups in question, treating him as an independent person in each of the groups.

The wage earners form a class of suppliers which it is especially important to include in the exchange service. The increase in the exchange of their "goods" means nothing less than a reduction in unemployment. It is of course only the unemployed who could form part of the exchange organization. A fully occupied workman has indeed no "goods" which he can deliver. It would not be practical that each unemployed person should be a personal member of the exchange organization. It would be more efficient if the various *trade unions* were members. When the total exchange table for the country had been balanced and the warrants issued, the trade union in question would then have to decide *which* of its members should start activity and thus get the warrants. To start with, it is of course very improbable that the balancing of the whole exchange matrix for the country could take place in such a manner that all the unemployed in all the trade groups received occupation.

13. The Exchange Requests and Their Balancing

The exchange could best be organized in definite periods, for instance by *months*. For the sake of simplicity I will in the following speak of the exchange period as a month, but the reasoning is quite general and applies irrespective of the length of time one finds it advisable to choose for the exchange period.

The first step towards planning the exchange in a definite month is the collecting of exchange requests from the members. In these requests the members would have to specify the amounts for which they desire warrants on the various groups of suppliers, and furthermore give a description of the commodities or services they propose to de-

liver. These requests would first have to be criticized by the district office or, if the case arises, by a district committee appointed especially for this purpose. It would have to be verified that the total amount which the member in question had requested was in a reasonable proportion to what he could be expected to be able to supply.

In the beginning, this preparatory criticism of the request would have to be made very strict. There could be fixed certain comparatively low maximum limits for the various categories of members. For an ordinary productive concern one should perhaps in the beginning maintain the total sum of goods warrants within say 20 per cent of the production capacity of the concern in question. For other categories the maximum limit could be fixed higher, for instance for unemployed workers virtually at 100 per cent. After the system had started to operate and experience had been gained, it would naturally be an easy matter to increase some of the limits.

The question of preliminary criticism of the exchange requests is connected with the question as to the manner in which one would formulate the legal claims which the National Exchange Organization got on the supplies by the fact that the Organization gives them warrants at the beginning of the month. More precise priority rules for these claims would have to be worked out.

The criticized and, if it should be the case, reduced exchange requests would have to be added up by the district offices, and the totals for for each individual group of suppliers would have to be sent to the national headquarters of the Organization. Here the figures for the entire country would have to be added up in order to get the total requests matrix for the country. The figures in this matrix would then have to be dealt with on some such principles as explained in the preceding Sections. On the basis of the balancing method which may have been adapted the necessary percentages, partakers' percentages, rationing coefficients, etc., would have to be computed and the final exchange figures for each group fixed. The total corrected exchange matrix thus obtained would then show complete demand—supply balance apart from the service taxes that may have been introduced.

Irrespective of the principles of balancing that had been chosen, it would be practical not to carry out the calculation of the necessary percentages to many decimal places, but only use the adopted balancing principle to compute a table that showed *approximate* agreement between demand and supply, and then make the final balancing of the exact amounts more or less by a rule of thumb.

This balancing of the exchange matrix should be done on a *national* basis. No attempt should, for instance, be made to make the exchange service a closed cycle within certain geographical spheres. It must be

sufficient that demand and supply for each group balance when taken as a whole for the entire country. To attempt a balance for any special geographical sphere would be in direct conflict with the fundamental idea underlying the exchange service. The aim should just be to arrange for a cooperation between the various parts of the country, and between the special branches of industry within the national border.

14. The Issuance and Utilization of the Exchange Warrants

When the national headquarters had balanced the exchange accounts for the various groups, the necessary number of exchange warrants would have to be printed. These should be sent to the district offices which must then see to the distribution to the individual members in the district. The national office would have to keep account of the amount of warrants issued to the various suppliers' groups in each district, and the district offices would have to keep the further detailed account of the amounts issued to the individual members in the groups. This latter accounting would presumably be done most easily in the form of loose leaf, the amounts could simply be entered in a special column on the original request blanks received from the members.

It would probably be advantageous only to issue warrants in a few denominations, e.g., only use warrants for \$10 and \$100. It is desirable to simplify the technical printing apparatus as much as possible because the printing of the warrants is something which—if the system acts according to intention—will take place each month.

Safe methods of auditing the accounts would naturally have to be arranged. In addition to the local auditing, which could be left to the district authorities to arrange, there would have to be organized an audit by travelling auditors directly under the National Headquarters. The danger of embezzling will in all probability scarcely be great in such a system where all service is to be *entirely balanced at short intervals.*

It must be a fundamental principle that one endeavors to make the use of the warrants as *simple* and *convenient* as possible. In various ways efforts must be made to make the warrants easily exchangeable so that there is created the least possible friction during the process whereby the warrants flow back to those who at the end of the month are again to deliver them up to the Exchange Organization.

In order to obtain this, the warrants ought to be *made out to bearer*, so that anyone who obtains possession of a warrant for the month in which it applies can use it. It means, for example, that townspeople may simply hand over to their grocer those of the warrants which apply to farm products. The grocer for his part would then use the warrants for his purchases. A certain grocer will perhaps be used to dealing with

a particular farmer at a particular place. There is naturally nothing to prevent him from using the warrants as payment just *vis à vis* of this farmer, on the assumption of course that the farmer is a participant in the exchange service during the month in question. Before putting through his order the grocer would probably ascertain that the farmer is a member. In this manner it would be possible to use without any alteration the whole technical exchange apparatus that already exists.

Another condition for the warrants being easily used in the manner indicated, is that they are not *drawn on* specified suppliers, but only on large groups of suppliers. Even if the warrants are drawn in this general manner there is every prospect that the exchange process will lead to such a result that each single individual supplier at the end of the month comes into possession of the quantity of warrants which he is to account for. The guarantee that this result will be attained lies in the fact that none of the participants has the right to acquire (through sales of his products) warrants for an amount *greater* than that which he was allocated at the commencement of the month. The most practical course would probably be simply to lay down the rule that any amount which a supplier may have acquired of warrants during the month over and above the amount which he was granted at the beginning of the month *shall be valueless*. That is to say, the National Organization undertakes no obligation to pay him in cash the countervalue of the surplus of warrants which he may thus hold.

This method of assuring the right distribution of the back-flowing warrants can be illustrated by the arrangement used when there are issued *non-numbered* tickets to a theatre. The method will not break down just because the tickets are not numbered. It only means that one leaves it to the individuals themselves to find their places. Of course this may result in some having slightly better places than others. Those who obtain the best places are simply those who take the trouble to arrive in good time before the performance commences. Every participant is, however, in the same position in this respect so that the arrangement contains no injustice. And, furthermore, it must be admitted that it is of subordinate importance whether some obtain a little better places than others, provided only that all the places are fairly good.

This naturally does not mean that one shall not do what one can in order that the process of adjustment can take place as easily as possible and give as great advantage as possible for those who use the warrants. In this regard the theatre illustration might be supplemented somewhat. If there is a big rush one would naturally take auxiliary measures in order to conduct the public to their places. One would establish queue formations, give out notices at short intervals as

to where seats are still available, etc. The main point is that when everybody knows that there is sufficient seating for everybody provided with a ticket, the question of getting people to take their places quietly will always be a problem which can be surmounted.

One of the things which will contribute towards a proper distribution of the back-flow of the warrants to the suppliers is that an easy opportunity be given for *voluntary clearing agreements* within the various branches. For instance, the shoeshops in a given city may establish a collegiate agreement to the effect that if the shop X in the course of the month has been able to acquire a greater amount in warrants than it is to account for, while on the other hand the shop Y has not been able to acquire a sufficient amount, Y shall have opportunity to buy a number of X's excess warrants *at a reduced price*. By such an arrangement X and Y would share the extra profit which X has obtained, because its exchange has increased by a greater amount than that which was originally the intention.

Also with regard to the distribution of the warrants which the participants receive at the beginning of the month, there should be open full access to voluntary clearings. Two workmen who have both received warrants but not exactly with the desired distribution may, for instance, exchange between themselves shoe warrants and warrants for farm products.

If the exchange service gets going on a really large scale, there would certainly develop such arrangements and many other forms of clearing activity. All this should not only be permitted but encouraged. It would do much to increase the effectivity of the entire system.

15. Information Service. The "Redeemability" of the Warrants

The National Exchange Organization should arrange a comprehensive information service regarding the system and its functioning. There should, at frequent intervals, be printed up-to-date lists of the suppliers within the various branches and within the various parts of the country. Alphabetical and systematical lists of the kinds of goods involved would also be useful. If the system once gets going well, advertizing by private firms would undoubtedly also help in this information activity.

All this would help the majority of people, who have warrants, to find out comparatively easily where and how they could use their warrants. The National Organization must, however, be prepared for all eventualities and therefore take special measures whereby everyone who has a valid goods warrant can be quite certain *under all circumstances* to obtain that for it for which it is made out. This arrangement is for the warrants what gold redemption is for the ordinary paper

notes. The National Exchange Organization must regard it as its duty always *to help the public* with the redemption. It must never permit itself to do what many central banks of issue frequently have done, namely to maintain redemption in name and carry on an actual sabotage *vis à vis* those who desire to avail themselves of it. If the National Exchange Organization lives up to this duty of always helping the public, the warrants will obtain a "redeemableness" which makes them immeasureably more secure than any of the usual kinds of money which are only based on gold.

Let us look at a few practical measures which the National Exchange Organization could adopt to ease the public's use of the warrants. Let us suppose that a person has a warrant but is late in utilizing it. The end of the month is perhaps drawing near so that the majority of the shops at the place have already used their quota. The person in question is of course certain that there actually exist suppliers who are prepared to deliver goods for his warrant. Indeed he knows that to each warrant issued there corresponds an obligation to deliver. The only problem for him is to *find* the right supplier.

For such cases the county offices and, if the case arises, the National office itself, ought to organize a special form of information service. The ideal would be that the National office keep a complete current survey of the progress of the activity within the various branches and in the different parts of the country based on current reports from the suppliers telling how much of their quota they had delivered up to that time. A speedy publication of these summaries, for instance through the daily papers, would greatly assist in directing the movement of the still floating warrants. Such an arrangement, however, would cause rather a large apparatus, which in the beginning should perhaps be avoided. To start with, one could perhaps content oneself with an arrangement whereby the suppliers who had not yet delivered their quotas, say, after the end of the first three weeks of the month, reported this. Or one could arrange reports on the basis of *estimates*.

16. Special Measures for Those who have not Succeeded in Placing their Orders

Even if the information service is fairly well organized, it is conceivable that some participants at the end of the month have still not found suppliers for their warrants. The simplest arrangement for the National Exchange Organization would then be to declare these warrants valueless. I do not believe, however, that this would be a wise policy. Even if it should cause a certain extra apparatus, one should do what one can in order to secure these "laggards" also. This will be

another feature of the National Exchange Office's endeavor to assist the public.

The following arrangement might be possible for the laggards. Those who at the end of the month have not yet placed their warrants should within a certain period, e.g., ten days, send in the warrants to the National Exchange Organization with a detailed *order* for the goods they desire. At the same time every district office should send a report of the amount of warrants which the participants had properly accounted for by the end of the month, and also a survey of the amounts which still remained open within the various branches.

Against these latter amounts the National Exchange Organization would now be able to draw *supplementary warrants*. These supplementary warrants should (in contradistinction to the original main warrants for the month) be drawn, not only on a group of suppliers but more specifically on such a group in a particular district. It is possible that they might even be drawn on a particular supplier; in this latter case the name of the supplier should be added by the district office inasmuch as it will scarcely be practical to send names of suppliers to the National office. The laggards would receive such supplementary warrants corresponding exactly in kinds of goods and amounts to the orders they had placed with the National Exchange Organization. And these supplementary warrants would now contain such an exact indication of supplier that no difficulty would be caused in finally obtaining delivery.

It is conceivable that, even after the issue of the supplementary warrants, there would be some participants who remain with a balance for which they had not handed in a settlement. Such balances would be of two kinds. In the first place it might be a balance with a participant who was capable of delivering but where customers had simply not appeared. Such cases will only arise if a number of warrants for one reason or another are *lost* on their way through the exchange machinery. The number of such warrants will scarcely be greater than the number of unfetched prizes in a money lottery. Where such cases arise the National Exchange Organization could simply close the account and dis regard the balance or, if it is found practical, make a certain deduction in the amount of warrants which the participant in question receives in the next month.

Another kind of unsettled account will arise in cases where one of the participants has failed and been put under receivership in the course of the month. There must then be present a direct case of fraud, as the person concerned must already at the beginning of the month have known his position. In such cases there will at the end of the month arise a corresponding number of *unredeemed* warrants (they will pre-

sumably be presented to the National Exchange Organization in the manner mentioned in the preceding Section). The National Exchange Organization must honor these warrants, if the case arises, by a direct payment in cash. And the National Exchange Organization itself must then prosecute its claim *vis à vis* the supplier and obtain what is possible of the claim. If the preliminary criticism of the requests is properly carried out, the loss to the National Exchange Organization on this score would be very small in comparison with the whole month's volume of transactions. Such losses must be covered in the same way as the ordinary administration expenses.

It should be a general principle that all effects of a transaction which has taken place in a particular exchange period shall be completely deleted *shortly after the end of that period.* Old accounts, old uncovered items, and the like, should not exist. Even if a number of small losses must be written off, it is better to make a clean sweep so that the Organization and its members are always concerned with the present services and counterservices. As part of this policy it should be laid down that the supplementary warrants issued by the National Office should only have validity for a certain shorter period, for example 20 days, and that thereafter all the items which must still remain open should be regarded as proscribed.

17. Connections between the Exchange Economy and the regular Money Economy

The activity which is set on foot by the exchange operations will in several ways affect the ordinary money economy. Even if the circle of participants in the exchange transactions is made very large, it is not likely that *all* activity will be carried on within this circle.

When a participant sets up his exchange request there is, of course, nothing to prevent him from reckoning that the carrying out of the exchange proposed may involve also certain *monetary* transactions for him. Some of these monetary transactions will perhaps take place between the participant in question and certain National activities which are outside the exchange circle, while others will take place between him and certain activities which affect importation or exportation. In both cases the exchange transaction will probably act to stimulate also that economic activity which takes place outside the exchange circle.

This will always be a benefit in so far as the national activities are concerned. With regard to the activities connected with the international exchange, it is conceivable that situations will arise where the stimulation of the activity will increase importation more than is desirable. This stimulation of importation is, however, a question which

it is scarcely correct to judge on the basis of the mechanism of the planned exchange service only. Importation is a problem which must be taken up for special consideration. If it should prove that it had in a given case been too great—either because of the ordinary money economy activity, or on account of the exchange service—an effort must be made to regulate the circumstance by the means which are known in this sphere: rise in customs duties, rationing of foreign exchange, import prohibitions, and the like.

Some simplified examples will illustrate the way in which a connection can be made between the exchange service activity and the ordinary money economy. Let us first consider a boot and shoe factory in Norway which uses Norwegian hides and employs Norwegian laborers only, and where no accessories are imported from abroad. Let us assume that the owner of the factory is granted Kr.1200 warrants for farm products and Kr.1000 warrants for Norwegian boot and shoe workers. Let us further suppose that of the Kr.1200 he uses Kr.700 to purchase hides which are used as raw material for the manufacture of boots and shoes, and that he further uses warrants for Kr.500 for the purchase of butter and eggs for his own housekeeping. The Kr.1000 warrants for Norwegian boot and shoe laborers he uses as wages. The sales value of the shoes we assume to be equal to the total of the warrants received, namely Kr.2200. Here the circle is entirely closed. The net result for the owner is that he has received Kr. 500 worth of butter and eggs. Apparently he has obtained it gratis. In reality it is, of course, not gratis. From an economic point of view it is an allowance for his contribution towards the result of the production process.

Next, let us modify the example to the effect that the above production costs of Kr.1000 does not in its entirety consist of wages, but that, for example, Kr.700 is wages and Kr.300 is depreciation on machinery. This machinery may have been imported or made in the country. This is of secondary importance in this connection, the main thing is that the machines represent a service from someone who does not participate in the exchange circle. In this case the Kr.300 amortization must appear in the manufacturers' ordinary money accounts. But this naturally does not prevent him from participating in the exchange transaction.

He can do it in two ways. We assume that he will produce the same quantities of shoes as assumed in the first example, that is, shoes valued at Kr.2200. He can then either reckon that of the Kr.2200 value of products only Kr.1900 is to be disposed of through the exchange service, while Kr.300 of shoes is to be sold in the usual manner for cash. In this case, in his request warrant he will demand: Kr.1200 warrants for farm products and Kr.700 warrants for shoemakers' work. Of this he will now, as previously, use Kr.700 for the purchase of hides and

Kr.500 for butter and eggs, which together amounts to Kr.1200 warrants for farm products. In addition he will pay out as wages Kr.700 warrants for shoemakers work.

He can, however, also reckon that the whole of the product value, namely Kr.2200, shall be disposed of through the exchange service. In this case he must add a new item in his request warrant. It must be an item which is not in connection with the footwear production but covers certain other expenditure which otherwise would have gone *via* his ordinary money economy. He may for example, want to have his house painted. Suppose that it would have cost Kr.300. By adding in his request for Kr.300 warrants for painting, he will obtain an exchange budget through which the whole of the product value, Kr.2200, can be disposed of. He will now ask for Kr.1200 warrants for farm products, Kr.700 warrants for shoemakers' work and Kr.300 warrants for painters' work. This exchange budget has the same effect for him as if the whole of the activity in the manufacture of boots and shoes—inclusive of amortization—was settled in and with the exchange transaction, while he had had Kr.300 cash outlay for painters' work. His profit on the transaction is naturally the same as in the first case, namely Kr.500.

18. Cooperative Construction Companies in Connection with the National Exchange Organization

The exchange service as it has been sketched in the foregoing does not apply only to articles of consumption. The system opens a certain opportunity for private concerns to use the exchange service as a link in the construction of permanent means of production and plant. When, for example, a building is to be erected, the builder can pay part of the wages in the form of warrants. This applies even if the constructural work is of such a kind that its result does not enter into the exchange system but must continue to be owned by the person who has erected it. Only in this case the person in question must carry on some *other* activity also, the products of which can enter into the exchange service. He must have certain products by the sales of which he can acquire warrants, (since he will scarcely have any interest in purchasing warrants for cash).

This naturally limits the importance of the exchange service for the actual construction activity. This fact is connected with one of the principles which formed the basis for the balancing of the exchange matrix, namely, that the participants should not do any *saving* by the exchange system. This would complicate the acounting considerably. This question can however be solved by the organization of a certain kind of concern: *cooperative construction companies* in connection with the National Exchange Organization.

By means of these companies the exchange service can be extended so as to embrace also such activities as those where the participants give work or goods, without immediately receiving in exchange other goods or services, but where they become participants in a saved-up value. It is probable that such an arrangement would be of considerable interest both to large and small savers, not least because the saving would be invested in *real values*, that would not suffer from monetary depreciation.

The cooperative construction companies could be organized in approximately the following manner: They must be under public control. Their financing should be based mainly on goods and services obtained through the National Exchange Service. The technique for the accounting in connection therewith would simply be that the construction companies, like other members of the National Exchange Organization, should send in their requests each month. As a counter-service, however, these companies would not offer to deliver other goods and services, but *property shares* in the company. The property shares could either be issued in the form of participation certificates or the participants could simply be credited on a current account with the company.

It would be a sound principle if these shares or accounts established through the National Exchange Organization were, practically speaking, the only financing basis for the construction companies. One should not perhaps definitely prohibit the construction companies from taking up loans in the usual forms (bank credits, bond loans, etc.) but there should in any case be fixed very strict limits as to how large a part of the total capital could be obtained in this manner. If capital should be obtained over and above that which is saved through the National Exchange Organization, it would be more reasonable that the State should enter as a share owner.

With regard to the economic exploitation of such enterprises distinction may be made between the following three cases:

a. The plants erected are of such a nature that they can be kept going on a regular profit basis. In this case they should be managed and exploited as private enterprises.

b. They are only partly rentable in a private economic sense (e.g., harbor constructions). In this case it would be reasonable that the State pay a certain *annual fee* to the construction company *for the social service* which the plant in question renders.

c. The plants are wholly unrentable in a private economic sense, that is to say, it is found impractical to institute such rules for their exploitation as would make them privately economically rentable (e.g., tolls on roads). In this case the State must pay an annual fee large enough to keep the plant on a paying basis. The principle here is the same as under (b), only now the *whole* of that which is necessary for upkeep and

interest would be regarded as a fee payable by the community for the use of the plant.

The main advantage of the arrangement here outlined is that the financing of the constructional works does not come over the State Budget, so that no new taxes or Government loans are involved.

19. Some Further Possibilities

Many city and county expenses are of such a kind that they can be paid in warrants. This applies particularly to small communities in country districts. Here the use of warrants will in many cases be a clear advantage. And even in large city municipalities undoubtedly certain kinds of payment could, without inconvenience, be made in the form of warrants.

A portion of the *municipal taxes* could then also be paid in warrants. In reality this means that a system is effected where the citizens, to a certain extent, pay their taxes *in the kinds of goods and services which they themselves produce.* In many cases this would undoubtedly contribute towards making taxation less burdensome. Such taxes paid through the Exchange Organization would resemble the old taxes *in natura* but the whole system would now be more elastic and more efficient.

If first certain municipal expenses were settled in warrants, it is also conceivable that the *banks*, to a certain extent, would let some of their payments take this form. In any case this could be done where the municipalities take up bank loans. To the extent to which the banks find they are able to use warrants in their payments they would also be able to accept payments of *debt* instalments in the form of warrants. This would be a help towards liquidating frozen credits.

The methods which are developed in the foregoing for balancing a closed national exchange circle can in principle also be applied to exchange between countries. International Trade has at present developed more and more in the direction of contingents and quota agreements. Such agreements between *two* countries have, however, only limited possibilities. In the international as well as in the national sphere it is the *indirect* exchange that brings the big advantages. Sooner or later the question of three-cornered or many-cornered contigent contracts will therefore probably arise. The technique of the request matrix seems to be the logical technique to apply to the situation.

20. Conclusion

1. The organization of a National Exchange service for a country seems, both on theoretical and practical grounds, to be possible. There can scarcely be any doubt that such an arrangement would help greatly to *break through* the obstacles of a purely circulation kind which the

depression has created. Pessimism and lack of confidence would not be able to stop the exchange within such a system inasmuch as the participants would be placed face to face with the *fait accompli* that they *have* those things which will enable them to buy the goods they want. They would not first have to obtain the means by increased sale of their own products. The vicious phase connection between sales and purchases would be broken.

2. The arrangement involves practically no *financial responsibility for the State.* The State's expenses will only be for administration and to cover losses caused by direct frauds.

3. The arrangement is enormously *elastic.* According to the desires of the participants, the volume of the exchange may be increased or decreased, or, if the case arises, may be diverted in new directions, without necessitating any great alteration in the administration and the technical apparatus.

4. If the system gradually works itself in, one will thereby obtain a real *survey* of the forces that exist both on the demand and supply side in economic life.

To be concluded in an early issue

AVIS DE LA RÉUNION DE LA SOCIÉTÉ D'ÉCONOMÉTRIE À STRESA, LAGO MAGGIORE, ITALIE, SEPTEMBRE 1934.

La 4e réunion européenne de la Société internationale d'Économétrie se tiendra à Stresa, Lago Maggiore (Italie), vers le 25 septembre 1934.

Le programme des travaux sera arrêté, au cours de l'été, dans le même esprit que les prédécents, par le Comité permanent présidé par Monsieur Bowley.

Les membres qui envisagent de participer à la réunion sont priés de le faire connaître dès que possible à Monsieur G. Del Vecchio, 8–10 via delle Lame, Bologna (Italie), qui a bien voulu se charger de recueillir les adhésions et leur fera parvenir les circulaires ultérieures.

Ils sont priés, en outre, de faire connaître s'ils désirent participer à un arrangement collectif pour le logement et les repas, en vue de réduire les frais de séjour.

Les adhésions devront être accompagnées du versement de vingt lires pour la couverture des frais d'organisation.

Les membres désireux de proposer une communication devront en adresser le texte ou tout au moins une analyse assez détailée à Monsieur R. G. D. Allen, London School of Economics, Houghton Street, Aldwych, London, WC 2, le plus tôt possible.

Le Vice-Président
F. Divisia

5 Mai 1934

ADDRESS OF THE ECONOMETRIC SOCIETY TO THE UNIVERSITY OF LAUSANNE

THE following address has been drawn up and sent to the University of Lausanne on the occasion of the centenary of the birth of Léon Walras which occurs this year:

> *A l'occasion du centenaire de la naissance de* LÉON WALRAS, *les membres soussignés de la Société Internationale d'Économétrie tiennent à exprimer leur reconnaissance à* l'UNIVERSITÉ DE LAUSANNE, *pour le service qu'elle a rendu à la Science, en dotant d'une chaire l'enseignement de la théorie de l'équilibre économique, et en lui imprimant un éclat qui a largement contribué à son rayonnement dans le monde."*

The address was signed by 317 members of the Econometric Society from the following 29 countries:

Argentina	3	Holland	12
Australia	2	Hungary	8
Austria	11	India	4
Belgium	1	Italy	33
Bulgaria	2	Japan	3
Canada	2	Jugoslavia	1
Chile	2	Norway	4
China	9	Philippine Islands	2
Czechoslovakia	6	Poland	7
Denmark	8	South Africa	2
Egypt	3	Spain	1
England	17	Sweden	7
Finland	1	Switzerland	8
France	33	United States	109
Germany	16		

Irving Fisher
President

LÉON WALRAS[1]

By J. R. Hicks

I

Like John Stuart Mill and John Maynard Keynes, Léon Walras was the son of an economist. His father, Auguste Walras, was one of those excellent people (they seem to have existed since very near the dawn of history) who taught the true but unhelpful doctrine that value depends on scarcity (*rareté*);[2] the son followed the father's teaching, but added to it something which lifts it on to another plane of precision. He defined *rareté* as *l'intensité du dernier besoin satisfait par une quantité consommée de marchandise*;[3] scarcity equals marginal utility.

His position with Jevons and Menger as one of the independent discoverers of the Marginal Utility principle is generally regarded as Léon Walras' chief title to fame; and this no doubt justly enough. But anyone who comes a little closer to these writers cannot help feeling a little resentment at the habit of classifying them together, even for the joint receipt of such an honorable title. For each of them made contributions to economics which are peculiarly his own, and it is for these special contributions that they are still worth reading today.

Indeed, the modern reader of Walras' *Éléments d'Économie Politique Pure* is struck by its affinity, not with the work of Jevons or Menger, but with that of Marshall. For a quite considerable part of the way Walras and Marshall go together; and when they separate, it is a difference of interests, rather than of technique, that divides them. While Walras was seeking for the general principles which underlie the working of an exchange economy, Marshall forged an analytical instrument capable of easier application to particular problems of history or experience. Yet, since the followers of Walras cannot always afford to be pure philosophers, and Marshallians have their moments of reflection, the two systems have inevitably tended to grow back into one another as the years pass by.

This affinity between two writers of different upbringing and obvi-

[1] Born at Évreux, 1834; in 1870, after ten years of journalism, business, and other miscellaneous activities, he became Professor of Political Economy at Lausanne, a position which he held until 1892. His *Éléments d'Économie Politique Pure* appeared in two parts: the theory of exchange in 1874, the theory of production in 1877. (Second edition 1889, 3rd 1896, 4th 1900.) He died in 1910.

[2] A. A. Walras, *De la nature de la richesse et de l'origine de la valeur* (1832).

[3] *Éléments d'Économie Politique Pure*, p. 76. All references to the *Éléments* will be to the *Édition définitive*, i.e., fourth and subsequent editions.

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ously very different mental outlook—their simultaneous development of what was then a very new line of thought—looks at first sight surprising, and one feels almost obliged to explain it by the intrinsic excellence of the path they followed: "it seems no honest heart can stray." Yet in fact there is a clear historical reason for it, one decisive influence we know to have been felt by both. Each of them had read Cournot.

Now although each makes a specific acknowledgement to Cournot, it is in each case couched in very general terms.[4] They each tell us that Cournot showed them how to use the differential calculus in economics, and this may mean much or little. But it is at least striking that certain very significant elements of Cournot's mathematical economics, going far beyond the mere idea of using mathematical methods, appear in Walras and appear in Marshall.

One of these is of course the demand curve itself (which already implies a resolution to treat economic quantities as if they are continuous variables). But more important, and less obvious, is the conception of perfect competition. Cournot's analysis, it will be remembered, passed from Monopoly to Duopoly (or Limited Competition), and from Duopoly to Unlimited Competition, which he defined as a state of affairs in which no single producer is able to influence appreciably the prices of the market. It was this last conception (applied to the theory of exchange value generally) which enabled Walras and Marshall to overcome the difficulties which had baffled Jevons, those difficulties which arise from the differences in the wants of different buyers of a particular commodity.[5] In the hands of Walras, this conception of perfect competition was converted into a special technique of using prices

[4] Walras, *Éléments*, Preface, p. viii. Marshall, *Principles*, Preface to the first edition, pp. ix–x.

[5] Cf. Jevons' ugly and unsatisfactory device of "trading bodies," which smudges over the distinction between monopoly and competition.

The relation between Jevons' work and that of Cournot is curious. When he wrote the first edition of his *Theory* Jevons had not read Cournot; but he had read Lardner's *Railway Economy*, "which treats certain questions of Political Economy in a highly scientific and mathematical spirit. Thus the relation of the rate of fares to the gross receipts and net profits of a railway company is beautifully demonstrated in pp. 286–293 by means of a diagram. It is proved that the maximum profit occurs at the point where the curve of gross receipts becomes parallel to the curve of expenses of conveyance." (*Theory of Political Economy*, 1st edition, pp. 17–18.) Lardner thus plotted total receipts and total costs against price—a peculiar way of putting it, to our ideas; but it is obvious that in so doing he adopted the most direct way possible of expressing the fifth chapter of Cournot's *Recherches* in geometrical form. Whether this is really what happened, I am unable to say; all that can be said is that it is definitely possible. For at the time Lardner wrote his book (1850) he was living in Paris, and so was Cournot; and there was at least this link between them, that in 1835, three years before

as economic parameters. Although of course this technique was used by Marshall as well, its very consistent employment is highly characteristic of Walras' work.

With this equipment, it was fairly easy to give an adequate analysis of the simple exchange of two commodities under competitive conditions. (Cournot had confined himself to the selling of products by producers, and did not examine the logically prior problem.) Accordingly, we find Walras beginning his *Éléments* in this way (1874), and Marshall following with a substantially equivalent analysis, hidden under the guise of a theory of International Values (1879).[6]

Walras' treatment fails of complete generality in only one respect; the downward slope of the demand curve is not quite so inevitable an assumption as he thought it. But he was well aware that the downward slope of the demand curve does not necessarily imply that the supply curve derived from it is upward sloping. If a person is buying X, and giving Y in exchange, then, if his demand for X becomes inelastic, his supply curve of Y will turn back towards the price-axis. In this case it becomes possible that the demand and supply curves for Y may cut several times; but some of these intersections will be points of unstable equilibrium.

Faced with this difficulty of multiple intersections, Marshall cut the knot by his distinction between the "theory of International Values" and the "theory of Domestic Values." In "International Values," the possibility of negatively inclined supply curves is serious; but they are unlikely to be particularly important in practice, because the competition of domestic industry generally suffices to keep a country's demand for imports in terms of exports fairly elastic.[7] In the theory of "Domestic Values," we may take commodities to be usually sold by producers or dealers who have themselves no direct demand for what they sell. Negatively-inclined supply curves can then only arise from increasing returns.

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he wrote the *Recherches*, Cournot had translated a book on Mechanics by Lardner into French.

But if we cannot prove the filiation, so much at least is clear: Jevons started from a theory of monopoly substantially identical with Cournot's. Where Walras and Marshall had the advantage over him was in their possession, not only of Cournot's theory of monopoly, but also of his theory of Unlimited Competition.

[6] *The Pure Theory of Foreign Trade.* There is, I think, no question that Marshall's analysis is quite independent of Walras'. Yet they differ in only two ways: (1) Marshall uses aggregate curves, instead of the simple price-quantity curves used by Walras; (2) Marshall's "Increasing Returns" complication has of course no counterpart in Walras' simpler problem.

[7] This is, indeed, to interpret the early Marshall by the late. (*Money, Credit, and Commerce*, pp. 351–352.)

Apart from the reference to increasing returns (a problem he never seriously examined), this reliance on sale by producers, whose reservation demand is negligible, was Walras' way out also.[8] But before coming to that point, he widened the problem by a consideration of multiple exchange, where more than two commodities enter into the picture. In order to treat this question, he supposed one of the n commodities to be chosen as a standard of value (*numéraire*), in which prices are reckoned, but which is itself subject to no demand other than that which arises from its ordinary properties as a commodity.[9] There thus remained $n-1$ prices to determine. From the conditions of given stocks at the commencement of trading, and equalization of the marginal utilities of expenditure in all directions, he derived each individual's demand (or supply) for each commodity. Then the ordinary equations of supply and demand in each market give the conditions of equilibrium. They are n in number, but that in the market for the *numéraire* is superfluous, as it follows from the rest. There are thus $n-1$ equations and $n-1$ unknowns; a set of prices must therefore exist which satisfies the conditions of equilibrium.

Here, for the first time, we have a characteristically Walrasian doctrine. What is it worth? On our estimation of it our view of Walras' individual contribution to economics must largely depend.

Now it is, of course, quite clear that, even when they are applied to this pure problem of exchange, the equations are far too complicated to be of much use in analysing any actual situation. But that is surely not their function. Where they are supremely useful is in elucidating the general way the price-system works, and in giving us a classification of those factors which may be relevant to any particular case. In practice we have to select out of that over-long list those which are most important for each special problem. When that selection is performed, we may get a result which conforms to the simpler scheme employed by Marshall; but on the other hand we may not.

The types of equations used by Walras in determining exchange equilibrium are two; those which express the dependence of the amounts

[8] *Éléments*, p. 163. It may be questioned whether Walras had as much right to it as Marshall (often) had.

[9] "Nous supposons ici les achats et reventes de (A) comme intermédiaire s'effectuant de manière à n'influer en rien sur le prix propre de cette marchandise. Dans la réalité, les choses se passent tout autrement. Chaque échangeur a par devers lui un approvisionnement de monnaie en vue de l'échange et, dans ces conditions, l'emploi d'une marchandise comme monnaie, a sur sa valeur une influence que nous étudierons plus loin." (*Éléments*, p. 156.) Recent economic thought has suffered, I think, by its neglect of this valuable device. The *numéraire* is a fanciful notion, perhaps; but it is the only logical way by which we can suppose exchange (or lending) to take place *in natura*.

demanded and supplied by particular individuals on the system of market prices, and those which express the equality of demand and supply in particular markets. These two classes stand on very different footings. So far as the first class is concerned, they have become the essential foundation for the whole branch of economics to which they refer. On them is based, and had to be based, all the work in the field of demand and of related goods, which has been carried out by Edgeworth, Pareto, and others. In the process of development Walras' conception of utility has been much refined; but we still work with Walras' equations, however differently we write them.

The second class, which expresses the equation of supply and demand in the different markets, seems much more simple and obvious; yet it has proved much more open to criticism, For it is on this class that the meaning of Walras' system of general equilibrium depends, and by far the most important divergence between Walras and Marshall turns on this point.[10]

Walras' own account of the nature of equilibrium is this. Persons come on to the market with certain stocks of commodities, and certain dispositions to trade ("dispositions à l'enchère") and a particular set of prices is proposed. If at these prices supplies and demands are equal, then there is equilibrium straight away. But if demands and supplies are not equal, prices will be changed until equilibrium is reached.

What, however, Walras does not make really clear is whether any exchanges do or do not actually take place at the prices originally proposed, when those prices are not equilibrium prices.[11] If there is no actual exchange until the equilibrium prices are reached by bidding, then Walras' argument is beyond reproach on the score of logical consistency, though it may be called unrealistic. (The market then proceeds under Edgeworth's principle of "recontract," or provisional contract.) But if such exchanges do take place, then, in general, the final equilibrium prices will be affected by them.

Marshall's way out of this dilemma was to concentrate on a particular market, where he could show that if the marginal utility of one of the commodities exchanged could be treated as constant, then the final rate of interchange would be independent of the path followed to

[10] Cf. Edgeworth's review of Walras in *Nature* (1889) and his controversy with Bortkiewicz in the *Revue d'Économie Politique* (1890–91). Also his comment in *Papers*, II, 311.

[11] "Les marchés les mieux organisés sous le rapport de la concurrence sont ceux où les ventes et achats se font à la criée, par l'intermédiaire d'agents tels qu' agents de change, courtiers de commerce, crieurs, qui les centralisent, de telle sorte qu'aucun échange ait lieu sans que les conditions en soient annoncées et connues et sans que les vendeurs puissent aller au rabais et les acheteurs à l'enchère." (*Éléments*, p. 44.) This remains ambiguous.

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reach it.[12] But this solution—which is, after all, only a very particular solution—is usually not available in the case of General Equilibrium.

Neither Walras nor Pareto faced up to this difficulty; when we do so, it is impossible to avoid the conclusion that the "Lausanne equations" are of rather less significance than they imagined. The equations of Walras are not by any means a complete solution of the problem of exchange; but they remain a very significant step towards such a solution. For Walras' system of prices will be reached, either if contracts are made provisionally or (a more important case) if people come on to the market on successive "days" with the same dispositions to trade, and there is no carry-over of stocks (or a constant carry-over) from one day to the next. When it is understood in the last sense, the theory of static equilibrium of exchange takes its place as a step towards the development of a complete theory with which future exposition is unlikely to dispense.

II

From the General Equilibrium of Exchange, Walras passed to the General Equilibrium of Production. For him, as for the Austrians, the problem of production fell into two parts: one relating to the pricing of factors of production, which are only used in combination with one another; the other relating to the rôle of time in production—the theory of capital.

The first of these problems (which corresponds to the Austrian theory of imputation) is really no more than an extension of the theory of value: it studies one particular kind of interrelation of prices. In this field Walras' original work was chiefly confined to a consideration of that problem which from his point of view is the simplest (though the Austrians naturally found it the hardest from their standpoint); the case where the "coefficients of production" are fixed, so that the quantities of all factors needed to produce a unit of each kind of finished product are technically given.

With fixed coefficients, and with perfect competition, the equilibrium prices of the products must depend on the prices of the factors; thus, given the prices of the factors, the whole price-system (of products and factors) can be derived by simple process of addition. But, given this whole price-system, the demands for products and the supply of factors can be determined from the tastes and abilities of the individuals composing the economy. Again, once the demands for the products are determined, the demands for the factors can be technically deduced. We can thus write both the demands for the factors and the supplies of the factors as functions of the set of factor-prices; and determine

[12] Marshall, *Principles*, Book v, Chapter 2; also the Appendix on Barter.

equilibrium in the factor markets as before. The equilibrium prices in the factor markets now determine the equilibrium prices of the products.

This solution is of course valid only under the assumption of fixed coefficients; but Walras was quite aware that it could easily be extended to the more realistic case of variable cofficients.[13] One cannot help thinking it to be a great pity that he did not trouble to work out this hint, for it would have led directly to the general law of marginal productivity.

Nevertheless, even as it is, this part of Walras' work has great merits. The particular relation which it exhibits has quite general significance, and could hardly have been discovered in any other way than this. Even when the coefficients of production are variable, so that a rise in the price of one particular factor influences the demand for it mainly by encouraging a substitution of other factors within industries, there will still be present this further tendency: that the factors which cooperate with this first factor will find it more profitable to devote themselves to the production of products for which relatively little (or none at all) of the first factor is required.

Again, we have here an excellent illustration of the value of Walras' work for the clearing up of questions of principle—the sort of question which Marshall so frequently left rather confused. Walras' equations give the most exact version that has ever been given of the "opportunity cost" element in value; and at the same time they preserve the essence of the "real cost" principle for which Edgeworth and Marshall contended. They exhibit the supplies of the factors as variable, but as determined by the system of prices in fundamentally the same way as the demands for commodities, with which they are interdependent.

It is hardly necessary, at this date, to discuss at any length that one of Walras' conditions which was so vehemently attacked by Edge-

[13] "Nous supposons, comme on voit, les coefficients . . . déterminés *a priori*. En réalité ils ne le sont pas: on peut employer, dans la confection d'un produit, plus ou moins de tels ou tels services producteurs, par exemple, plus ou moins de rente, à la condition d'y employer moins ou plus de tels ou tels autres services producteurs, par exemple, moins ou plus de profit ou de travail. Les quantités respectives de chacun des services producteurs qui entrent ainsi dans la confection d'une de chacun des produits sont déterminés en même temps que les prix des services producteurs, par la condition que le prix de revient des produits soit minimum." (*Éléments*, p. 212.) This passage first appeared in 1877. (The condition of minimum cost follows from that of maximum profit—under conditions of perfect competition.) For Walras' later work on Marginal Productivity, see particularly his "Note sur la réfutation de la Théorie anglaise de fermage de M. Wicksteed," which appeared as an appendix to the third (1896) edition of the *Éléments*, but was subsequently omitted.

worth[14]—the condition that prices equal costs of production, so that the entrepreneur makes "neither profit nor loss." For this device, in spite of its paradoxical appearance, is nothing else than the reckoning of "normal profits" (the profits which the entrepreneur could earn in other activities) into costs; and similar forms of definition are now adopted for their extreme convenience by many economists who would acknowledge no direct debt to Walras.[15] It may indeed be questioned whether the full implications of this method of statement have been explored—particularly with respect to its application to dynamic conditions. But the device itself needs no defence nowadays.

III

Those parts of Walras' doctrine which we have hitherto considered are on the whole uncontroversial; it is true that they raise difficult problems of interpretation, but no one seems to doubt that in some sense they are valid enough. It is these parts which have passed into the body of economic teaching; and when we want to study them we are inclined to go, not to Walras' own works, but to the rather more elegantly stated versions of his successors, such as Pareto or Wicksell.

Walras' theory of capital, however, has not reached this happy position. By Pareto it was simply ignored; by Wicksell it was attacked.[16] It has therefore not passed into any recognized "Lausanne" tradition, and is liable to be dismissed as something of an aberration. In spite of this, it has its merits; though there can be no question that it needs a good deal of repair in details before it can become a usable theory.

If a reader who is acquainted with the work of Böhm-Bawerk[17] and Wicksell approaches Walras' theory of capital, the first thing which will strike him is that it is purely a theory of fixed capital. Walras begins from a discussion of the capital value of income-yielding goods. He shows that the ratio of capital value to net income yielded (after (allowance for depreciation and insurance) must tend to equality for

[14] Edgeworth, *Papers*, I, 25.

[15] E.g. Robinson, *Economics of Imperfect Competition;* Keynes, *Treatise on Money.*

[16] Wicksell, *Uber Wert, Kapital und Rente*, pp. 142–3. Barone ("Sopra un libro di Wicksell"—*Giornale degli Economisti*, 1895) replied to some of Wicksell's criticisms, and apparently convinced him that he had overstated his case. Wicksell's comments in his *Lectures* (English edition, I, p. 171) are appreciably milder; while in his late paper "Professor Cassel's Economic System" (reprinted in the English edition of the *Lectures*, p. 236) he takes what I should consider a very balanced view.

[17] Walras' theory is of course earlier in date than Böhm-Bawerk's; it was substantially complete by 1877.

all such goods; otherwise people would sell the more expensive (relatively to yield) and buy the cheaper. Thus there emerges a "rate of net yield" (*taux du revenu net*), which, in equilibrium, must be equal for all capital goods.

How is the "rate of net yield" determined? By the condition that the prices of new capital goods must equal their costs of production. Granted that a certain amount of new saving is coming forward, this saving will give the demand for new capital goods.[18] The saving has then to be divided among the various capital goods that can be produced in such a way as to maximize the rate of net yield.

Substantially, that is Walras' theory; it is a theory, which, if taken literally, is open to very serious objections.

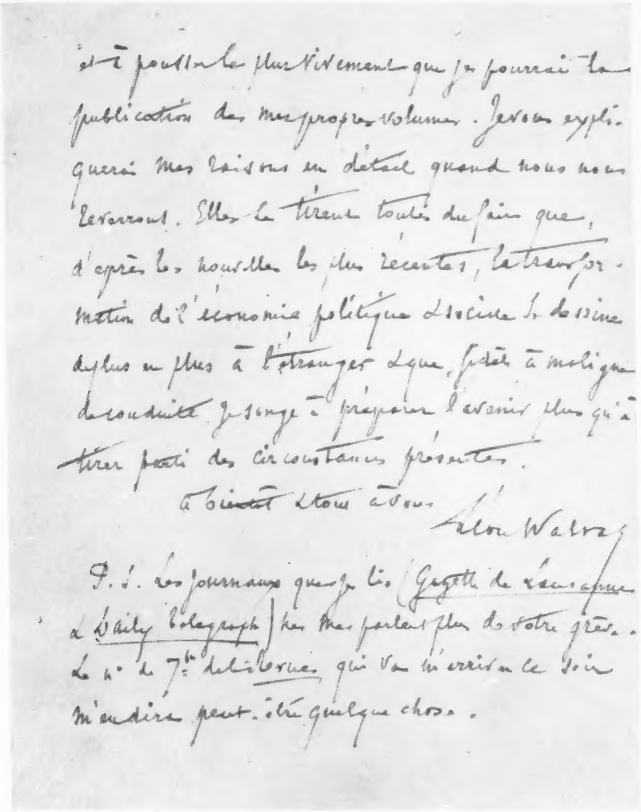
For one thing, as Wicksell pointed out, it determines the rate of interest on the market for new capital; and is therefore apparently inapplicable to stationary conditions, when no net addition to the capital equipment of the community is being made. Further, as Walras would have realized if it had not been for his confusion about the exact meaning of equilibrium,[19] it is only in a stationary state that we can get any sensible sort of equilibrium, so long as people expect the prices of products to remain unchanged in the future (as Walras tacitly assumes they do). This dilemma is fatal to the theory as Walras presents it.

But it is not necessarily fatal to the whole method of approach. For once we assume that the reinvestment of depreciation allowances is not technically given (in the way Walras supposed), but that these funds are reinvested according to the best prospects open for them at the moment of reinvestment; then the "new capital goods" become not only net additions to the capital stock, but also replacements, and the demand for these goods is no longer confined to new savings, but consists of depreciation allowances as well. With this slight extension, Walras' system becomes immune from Wicksell's criticisms; the capital market does not disappear in the stationary state.[20]

[18] Savings are of course also a function of the rate of net yield, which now enters into the determination of expenditure on the same footing as the prices of commodities. It must be remembered that savings are expressed in *numéraire*.

[19] The confusion we discussed above. It gets palpably worse in the later part of Walras' work. See, for example, the rather pathetic passage on pp. 214–215 of the *Éléments*.

[20] It is interesting to observe that, once this amendment is made, the limitation due to Walras' concentration on fixed capital disappears. For the method of reducing fixed capital to circulating, introduced by Jevons and Böhm-Bawerk, works both ways. If a machine is economically identical with a collection of half-finished goods which will be ready at different dates, so is a collection of half-finished goods economically identical with a machine.



Specimen of Walras' Correspondence

Walras did not make this amendment, but its possibility deserves attention; for it shows the essential rightness of his method, which survives the imperfect way in which he used it. Once the amendment is made, Walras' theory of capital becomes as good as Wicksell's, and better than Böhm-Bawerk's. It is still subject to the static limitations within which their theories are also confined, but it is as good a basis for extension in a dynamic direction as theirs—and in some ways it is perhaps better.[21]

IV

Walras' work on the theory of money,[22] and his relatively uninteresting writings on applied Economics, cannot detain us here. It was in pure economics that his real interest lay, and the discovery of the conditions of static equilibrium under perfect competition was his central achievement. Like many pioneers, he was a little vague about the exact meaning of some of his results, and was perhaps inclined to claim for them more than they are actually worth. Yet our consciousness of its limitations should not blind us to the greatness of his achievement. Static equilibrium is far from being the whole of economics, but it is an indispensable foundation; and the greater part of that foundation was laid by Cournot and Walras. There are very few economists who have contributed so much to the permanent body of established truth as Walras did.

Comment may be made in conclusion on two qualities of his work taken as a whole. One is the realization of the unity of economic life which emerges so forcefully from his pages. Other economists had had a sense of this unity, but none before had shown it so well. For the unity which Walras demonstrated is not a unity of resources being allotted among a single system of ends—the only unity which really appears in Menger—it is a unity of diverse individual ends reconciled through the mechanism of the market. Yet this unity is as real as the other. In a free economic system, under perfect competition,

> thou canst not stir a flower
> Without troubling of a star.

The other great quality of Walras' work to which we may here allude is its rigorous "methodological individualism." Far better than any earlier economist—better even than Marshall—he realized that the

[21] The "original factors" of the Austrians, being largely "bygones," are a thorough nuisance in economic dynamics.

[22] See Marget, "Léon Walras and the Cash-Balance approach to the Problem of the Value of Money" (*Journal of Political Economy*, 1931). In this field Walras did at least make a serious attempt to integrate monetary theory with the rest of economics; he did something to prepare the way for Wicksell.

only economic explanation of a phenomenon is its reference back to individual acts of choice. Even he did not emancipate himself entirely from that sham utilitarianism which was the bane of his contemporaries, and which led them to suppose that the working of the free market "maximized utility" for the community as a whole.[23] But this in his work is a mere excrescence, and is easily disregarded. In his central doctrines he held firmly to the true significance of economic subjectivism, and therefore broke with the Labour Theory of Value more drastically than Marshall, and quite as drastically as the Austrians.[24] For him individual choice was all-important in its function as explanation; and it is our realization of this which has led us to understand that it is not, for the economist, necessarily anything more.

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[23] Cf. Wicksell's final exposure of this fallacy (*Lectures*, I, pp. 73 ff.).

[24] If he was less conscious of this principle than they were, he wove it even more tightly into the structure of this theory.

ANNUAL SURVEY OF STATISTICAL INFORMATION: FAMILY BUDGETS

By Hans Staehle

The econometrician is often hampered by the absence of quantitative information, in spite of his efforts to restate problems of economic theory in such terms as will fit them for verification by actually existing statistics. The kind and amount of statistics available are thus very definitely limiting factors in some econometric works. But it may also be that the existing statistics invite theoretical enquiry, so that statistical information may occasionally be considered as a factor making for extension and progress of econometrics. For instance, whereas, in the field of cost theory, the state of statistical information is primarily a check on the progress of econometric studies, family budget statistics rather seem to be a branch of information which is full of promise. Dealing with some of the more *terre à terre* questions concerning family budget enquiries, an attempt will be made in the following to consider some of the many possibilities which this field of statistics seems to offer to the econometrician, provided he is patient enough to consider his assumptions carefully.

I. SOME OF THE TECHNICAL DIFFICULTIES INHERENT IN FAMILY BUDGET STATISTICS

Family budget enquiries are accounts—generally non-periodic—of the expenditure incurred on the average by a group or groups of homogeneous and representative families in a unit of time. Simple as this description appears to be, it covers a considerable variety of information. In fact, it may be said without exaggeration that there are hardly two independent enquiries which supply the same kind of data, and in spite of the efforts made by statistical bodies, particularly international ones, to provide for at least a certain amount of comparability in this branch of statistics, every new forthcoming enquiry is presented in a different way, and gives details which are different from those supplied on previous occasions or in other countries.

These differences may be classified into such as concern first, the general plan of the enquiry, second, the kind of information collected, and third, the presentation of the results.

The outstanding questions in the first group relate to the choice of the families to be included in the survey, with respect to their number, social class, occupation, location and (possibly) size; the period to be covered; and other more technical aspects, such as the method of collecting the information (e.g., through household books, or field

agents, or both), which are of course most relevant in considering the reliability of the outcome, but do not raise any great theoretical problems.

Opinions as to the number of families to be covered are divided. Sometimes the inclusion of relatively small numbers is advocated, provided greater detail can be given, on the ground that more detailed information is more useful than large but less detailed averages. The contrary view is also held occasionally. But, of course, this is not a question to be decided in a general manner, and the particular circumstances and purpose of the enquiry must be taken into account. From the theorist's point of view, it is not so much the number of families that matters, but first the homogeneity, and second, the representativeness, of the resulting data. For this reason, he will generally prefer enquiries which are concentrated locally[1] and, for most purposes, limited to only one social class, and possibly one or at least similar occupations of the family's head; and in general he will be willing to give up some of the representativeness (in the sense of a vast scope) for a greater homogeneity. This preference for homogeneous rather than "representative" data has also a bearing on the period of observation; clearly, it will normally be more or less indifferent whether the averages are obtained from expenditure accounts which cover a month or a year, provided a month is long enough to include at least once for every family an expenditure on each of the items it normally consumes, and not so long as to cover any violent price changes. That is why it would be interesting to have not only yearly averages (a practice which is becoming more and more general), but also averages for shorter periods, even when the enquiry has lasted a whole year. In general, the shorter the period of enquiry, the larger the number of families should be.

The information to be collected and its presentation raise a great number of technical questions which, however, seriously influence the uses to which the material can be put theoretically. Only a few of these questions can be discussed in this brief survey.

The most elementary postulate the theorist has to make is that wherever possible the information should not only show expenditure on each item, but also its constituents: price and quantity.[2] The separa-

[1] For some purposes, such as the application of Frisch's translation method, separate results for different localities are, of course, important.

[2] It is much to be regretted that important enquiries which are well done otherwise do not fulfill this condition and therefore become practically useless for most theoretical purposes. The reason for this omission is that, in the construction of time index-numbers of prices (the weights of which are as a rule determined by household budgets), it comes to the same thing whether one

tion of these constituents is sometimes a difficult matter, mainly on account of what is to be considered as an "item" or "article". The list of items must be rather detailed so as to justify the computation of an average price and an average quantity for each item. But it cannot well go into such details as to offer an absolute guarantee that the quantities and prices refer to exactly comparable things. These difficulties will be somewhat reduced by limiting the enquiry locally, where the variation of qualities and grades for articles having the same description is likely to be smaller. But, nevertheless, it is certain that the actual articles consumed by families, however similar otherwise but having different incomes, are different, at least with respect to quality, even when their designation is the same. This finds its expression in the fact that the prices paid on the average for a given "article" generally increase with increasing income. In addition, there is, of course, the difficulty of the units in which the quantities (and the prices) are to be given. For many commodities, reasonable units are nearly always available or at least the less reasonable units (e.g., so many herrings, or tins of canned vegetables) can often be reduced to reasonable ones with sufficient approximation. But some commodities (such as clothing items) are such that it is impossible to state the "quantities" in any comparable unit; the same applies to services and thus to the greater part of the items usually included under the heading "miscellaneous." Rents also are a difficult item to handle.[3]

Fortunately, the great popularity which "Engel's law" enjoys among practical statisticians has ensured the regular supply of two important features of the family budgets: (1) the expenditure on foodstuffs as distinguished from other expenditure, which is again almost generally split into clothing, housing, heating and lighting, and "miscellaneous items," and (2) the grouping of the families according to income. But, while this classification according to income is generally adopted and carried through in all the main tables,[4] the classes chosen are often unsatisfactory. Principles according to which the classes should be formed do not exist; and it must, of course, be realized that it is impossible to establish any hard and fast rules in this respect. One difficulty for

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computes $\Sigma p_1 q_0 / \Sigma p_0 q_0$ or $\Sigma (p_1/p_0) p_0 q_0 / \Sigma p_0 q_0$; obviously, knowledge of expenditure is sufficient in the latter case.

[3] See in this respect Guye's vigorous attempt in *International Comparisons of Costs of Living, A Study of Certain Problems Connected with the Making of Index Numbers of Food Costs and of Rents*, International Labour Office, Studies and Reports, Series N, No. 20, Geneva 1934, Part II.

[4] The most regrettable exception is its omission in the food consumption tables of the U.S. enquiry of 1918–19: U.S. Department of Labor, Bulletin 357, Washington 1924.

instance is this: however carefully an enquiry has been planned, it must always reckon on losing a certain number of the families chosen at the outset (through negligence, death, etc.), and may therefore arrive in the outcome at a distribution of its material with respect to income, locality, social class, etc., which is quite different from the one originally intended. In these circumstances, it would appear that the only means of deciding what classifications the material admits is to study *ex post* the distribution of its main characteristic features, such as the expenditure on food (as was done by Bowley in the Report of the Working Classes Cost of Living Committee 1918,[5] where the standard deviation was used). One may for instance assume that it is desired to apply a given classification by income to *a number of localities;* then the ratio of the standard deviation and the average of the expenditures on food per "man" or "equivalent adult male" (or other unit of consumption designed to eliminate the influence of the size, and age and sex composition, of the families) will yield a suitable indication as to the number of localities which it would be possible to distinguish.[6] Apart from other useful purposes which the regular supply of the standard deviations in the reports on budget enquiries would fulfill, and on which we shall dwell later, it seems clear that a general adoption of this practice would considerably improve the reliability and the presentation of their results. It might of course be that, as a consequence, certain interesting but unreliable tables would no longer be forthcoming, but that would be rather a gain than a loss. It is therefore to be hoped that practical statisticians will adopt the rule that every average given in the enquiry reports should be accompanied by the standard deviation or some other measure of dispersion.

[5] London 1918, Cd. 8980.

[6] This does not mean to say that it is not necessary to take into account the numbers of families which will come under each heading of the double classification. But if the number in one particular income class in one locality is found insufficient, so that the number of areas (or localities) has to be decreased, then the size of the standard deviation relatively to the average of the food expenditures will indicate whether or not the grouping may still be considered as homogeneous. Obviously, when the localities or areas for which a classification by income is desired, are given, then, if the number of families coming under one particular income class is insufficient, the number of these classes must be reduced; in this case, however, since the expenditure on food is a function of income, the relative standard deviations of food expenditure will have to be considered together with the relative standard deviations of the incomes in each of the new classes. When neither the number of localities nor the number of income classes to be distinguished is given, so that both may be chosen freely so as to suit the data best, then it should be possible by an intelligent interpretation of the standard deviations obtained by various experimental groupings to arrive at averages which will be not only based on sufficient numbers, but also as numerous, and yet homogeneous, as the data permit.

Another point where it would be desirable to obtain at least greater continuity, if not a somewhat greater degree of uniformity, is the scale used in the various enquiries with the view to eliminating the influence of the differences in size, and age and sex composition, of the families. Apart from Belgium, where fidelity to the "quet" is apparently unshakeable, the general tendency is toward choosing for this purpose the "adult male" as a unit of consumption. Yet the scales used are nevertheless quite different. Thus, the age at which the male is considered as an "adult" is rather variable, and the scale according to which women are translated into "adult males" is far from being the same in all cases. These differences may be quite justified under different climates. It is equally justified, but introduces great complications into the calculations, e.g., of index numbers, to apply different scales for the relation between age and sex on the one hand and normal consumption on the other, not only to the different main divisions of the budget (e.g., food and clothing), but also to the various items belonging to one great division (e.g., different foods). In fact, if the quantities are to enter into a system of weights, it will make some, and possibly a considerable, difference whether the quantities are taken per unit of consumption or per family. This is rather awkward in intertemporal or interlocal price comparisons, where it is often necessary to use materials (such as the otherwise excellent pre-war enquiries conducted by the Board of Trade) where no computation per unit of consumption was given. It is questionable therefore whether much would be gained by too great refinements being introduced into these scales.

The most important rôle of the computation of "adult males" is not so much to make the quantities consumed comparable as to offer a basis for the classification according to income. Obviously, the same money income means quite a different real income to families which are large from what it does to families which are small. Therefore, it would be meaningless and yield highly unreliable results if the income classes were formed according to family income. And although it can never be hoped to find a scale which will entirely even out the influence of the varying size of the family upon its expenditure, it is certain that the application of some scale is necessary if the material is to be even approximately homogeneous.[7]

[7] Unfortunately, the income classification adopted in the enquiries conducted by the Board of Trade above mentioned, and many others, is based on family income only. The influence of this omission is less serious where the size of the family increases with increasing family income; that indeed indicates that the increased income is mainly due to the fact that the children are old enough to contribute to the income of the family, so that the varying size of the family will be partially taken care of simply by computing the income and the expenditure items per person.

II. SOME ECONOMETRIC USES OF FAMILY BUDGETS

We shall now turn to consider a few of the uses which the material may serve. Since, however, it would again be impossible to deal adequately with the whole field we shall dwell only on a few instances, emphasizing the relations between the material and the working hypotheses which are being made more or less explicitly in the theoretical analysis.

The main assumption which must regularly be made whenever it is intended to use budget materials for theoretical work is that, given the prices and the income, expenditure is distributed in such a way as to yield maximum satisfaction. It is clear, however, that this assumption is a rather difficult one to make. In fact, the prices which are assumed to be given are actually variable, and considerable variations may have occurred during the period of the enquiry; it seems hardly possible to obtain results for such periods as exclude all movements of relative and absolute prices. The difficulty arising from price differences in different localities and between different grades and qualities of a given article has already been mentioned. On the other hand, as regards incomes, the assumption made amounts to supposing the flow of incomes constant over the period of enquiry. That again is hardly true in reality, since spells of short-time or even complete unemployment are likely to occur, and even where the flow of income was actually or approximately constant, there may well have been some uncertainty regarding its constancy, so that the situation was not a purely equilibrium one. It may, of course, be argued that these inevitable deficiences of the data must not be taken too seriously, since the resulting averages will nevertheless approximate the equilibrium situations, and that the deviations from it will cancel out, if not for the individual family over time, at least for the average of the families. This however is not necessarily true. Indeed, as regards first the averaging of the expenditure and its components for the individual family, if during the period of enquiry the price of only one particular good has varied, then, since the average price paid by the family is the weighted average of the prices paid in the course of time (i.e., $\Sigma pq/\Sigma q$), the resulting average price and the corresponding average quantity actually are observations on the demand curve of that family, only if the demand curve is one of constant elasticity and has not shifted during the whole period. Even granting the first part of this last assumption, the second is not very probable, unless the flow of income has not only been constant, but also been expected to be constant; and only then could the recorded average price and quantity be considered as having equilibrium values.[8]

[8] The importance of constancy in the flow of income makes it all the more de-

It seems clear, therefore, that owing to price variation and irregular flow of income in time and to price differences in space and between qualities of the same good, essential conditions for the attainment of consumers' equilibrium are not fulfilled by the ordinary budget material.

There is the further question regarding the varying circumstances of daily life and of the complications arising from deferred payments and installment buying. Since it is after all only the most orderly and "economic" families who will normally carry on records of expenditure during a rather prolonged period, the above variations do not, in the writer's opinion, exclude the possibility that the budget data reflect the individuals' or the families' struggle for maximum satisfaction. Favorable, however, though this is for econometric studies, it reduces the degree to which the results are representative of the population, or class of population as a whole.

The argument that the deficiencies of the data are likely to cancel out on the average as between different families in one income or other group of the classification, so that the average of their expenditures may be considered as having equilibrium values with reference to the scale of preferences of "the average family," is intimately connected with the question whether it may be assumed that the tastes and habits of all the families are identical (or at least normally distributed, whatever that may mean). This last assumption must always be made, if it is intended to reach any general conclusions, e.g., as to the probable relation between the quantities consumed of a certain article and income, as has been attempted by Marschak for commodities, expenditure on which accounts for only a small fraction of total expenditure.[9]

But even where the tastes and habits of all the families included are perfectly identical, the results will not be valid in the short run. Assuming indeed that the actual income R of a given family is changed to another amount R', a certain period will have to elapse before it consumes the same things as a family having had that income R' for some time. Without speaking of frictions such as rent contracts which prevent a ready adaptation, numerous trials will be required before the process of assimilation to the new income is complete and maximum satisfaction is attained in the modified situation.[10]

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sirable that the results of enquiries covering one year should be made available for shorter sub-periods, as above suggested. Experiments might meantime be made with such enquiries as those for Poland, Czechoslovakia, and Hamburg, which have all been carried on for several consecutive years.

[9] See, however, on Marschak's method, Frisch's criticism in *Revue d'économie politique*, 1932.

[10] This is fully realised by Marschak. To quote an actual example, it is quite likely that one of the income groups distinguished among the salaried em-

The question is, whether all these assumptions are such that they must either be made light-heartedly, or else the whole analysis be abandoned. It seems to the present writer that there are two ways to deal with this matter: one, somewhat precarious, which often necessitates long roundabout processes, and yet may seem inconclusive, is to judge from the results backwards (i.e., if the result is plausible, to conclude that the assumptions involved cannot be foolish), and the other consisting in a preliminary study of the distribution of the individual observations around their averages. We shall first deal briefly with this last possibility, and then give an example of the former.

It has been attempted to show above that the knowledge of the standard deviation for the averages of the more outstanding items of family expenditure, together with a knowledge of the corresponding measures for the incomes included in each class would be valuable as an indication of the homogeneity of the data. The interest of the econometrician, goes however much farther. He would like to see standard deviations for every average price and average quantity consumed also. For instance, should the dispersion of the prices paid for one particular article in any particular income class be large relative to their average, this would at once be a warning that the assumption

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ployees' families in the Swedish enquiry of 1923 is a case in point. To judge only from the quantities of food consumed, the families having annual incomes of 1650–1925 Kr. per "man" purchased in a manner corresponding rather to a lower income, as is shown by the following table:

QUANTITIES OF CERTAIN FOODSTUFFS (IN KILOGRAMMES) CONSUMED ANNUALLY PER "MAN" BY SALARIED EMPLOYEES' FAMILIES HAVING ANNUAL INCOMES PER "MAN" OF

	Kr. 1375–1650	Kr. 1650–1925	Kr. 1925 and above
Average income	Kr. 1537	Kr. 1804	Kr. 2217
Number of families	99	69	49
Veal	15.2	15.5	20.8
Mutton	7.6	7.9	10.9
Pork	35.6	35.6	42.9
Butter	41.9	44.2	59.4
Margarine	20.1	23.1	17.8
Bread	157.4	138.6	156.4
Flour	202.3	223.1	201.9

The consumption of meats is almost the same as for the lower class; butter is but slightly higher, whereas margarine is consumed in larger quantities than in any other income group; the high consumption of flour, together with a comparatively low one of bread, are also characteristic of a low standard. Were it not for the other items, the latter two instances might be accounted for by concentration of these families in rural communities where home baking is normal. Probably, both location and lack of adaptation to income contributed to the particular behaviour of these families. It would be interesting to follow up this and other examples of consumers' maladjustments in detail.

of constant prices cannot be made without further enquiry. It may, of course, be that the "article" is ill chosen and covers two or more different qualities; perhaps the geographical distribution of the families also may be uneven. At any rate, articles for which the prices paid are too widely dispersed would not be suitable for special study, unless the width of the dispersion could be reduced by reclassification, as might be done, though with much labor, if the budgets for the individual families were available, as is the case e.g., in the recent Polish enquiries.[11]

If the variation of both average prices and of their dispersions is found satisfactory for a given article, then a comparison of the relative dispersions of the quantities of that article consumed in different income classes, account being taken of the corresponding relative dispersion of the incomes included in each class, would yield indications of some importance, e.g., as to the income-elasticity of demand for the article in question (Marschak's problem). For this purpose it is necessary to know whether the quantities taken by the various individuals whose curves are to be combined into a market curve are different, even if their incomes are similar, i.e., if they belong to the same income class. It can, in fact, be shown that the elasticity of the market function giving the dependence of quantity on price, or income, or any other independent variable, is at any point of the market curve the weighted average of the elasticities of the individual curves of which it is compounded, the weights being the quantities taken by each individual at that particular value of the independent variable.[12] Con-

[11] Some articles are bound to have a relatively wide price dispersion, on account of high transportation charges when the families are not concentrated locally, e.g., coal; the price of gas and electricity also varies greatly from town to town; perishables may have wide price dispersions, even when the enquiry covers only a small territory. Some increase of the relative price dispersion may be expected with increasing income. Finally, seasonal fluctuations play a considerable part, e.g. for eggs.

[12] The proof is as follows: Let the quantities q_0 and q_1 of a given article consumed by two individuals O and I depend on any independent variable x (say, price, or income) and

$$q_0 = f_0(x);\ q_1 = f_1(x)$$

be the functions describing this dependence, and write

$$f_0(x) + f_1(x) = F(x)$$

for the resulting market curve when they are compounded. The individual curves will have the elasticities:

$$\eta_0 = \frac{df_0(x)}{dx} \cdot \frac{x}{f_0(x)} \text{ and } \eta_1 = \frac{df_1(x)}{dx} \cdot \frac{x}{f_1(x)}.$$

The elasticity of the market curve then is

sequently, if the quantities taken by individuals having the same income are not the same, i.e., if the relative dispersion of the qualities is considerably larger than the relative dispersion of the incomes belonging to that class, the resulting market demand function cannot be one of constant elasticity, unless it is assumed (1) that all the individual elasticities are the same, in spite of the difference in the quantities taken, and (2) that each individual curve has a constant elasticity over its whole length.

By similar studies of each case on its own merits, given the particular problem and the material by the use of which it is to be solved, it will be possible to judge the necessity of making particular assumptions, or the possibility of doing without them. At any rate, however, it would greatly facilitate the work of the econometrician, and certainly improve the quality of the materials, if Bowley's first step in the direction of supplying some measure of dispersion together with each average could be adopted generally.

For certain other purposes it is necessary to make assumptions, beyond those above mentioned, which are far more difficult to admit without verification, or, still more serious, which are obviously not fulfilled by the materials. This is, e.g., the case in measurements of the "price of living," i.e., the problem of price index numbers. It is well known that several writers have arrived independently at the conclusion that the "price of living" (or the ratio between money incomes yielding equal amounts of total satisfaction) lies between Paasche's and Laspeyres' index formulae which represent the lower and upper "limits" respectively. All of these writers are unanimous in basing this conclusion on the following assumptions: (1) that the individual or individuals in question do maximize satisfaction, (2) that they must have

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$$\eta = \frac{dF(x)}{dx} \cdot \frac{x}{F(x)}$$

$$= \frac{d[f_0(x) + f_1(x)]}{dx} \cdot \frac{x}{F(x)}$$

$$= \frac{df_0(x)}{dx} \cdot \frac{x}{F(x)} + \frac{df_1(x)}{dx} \cdot \frac{x}{F(x)}$$

which becomes by factoring

$$\eta = \frac{df_0(x)}{dx} \cdot \frac{x \cdot f_0(x)}{f_0(x) \cdot F(x)} + \frac{df_1(x)}{dx} \cdot \frac{x \cdot f_1(x)}{f_1(x) \cdot F(x)}$$

$$= [\eta_0 \cdot f_0(x) + \eta_1 \cdot f_1(x)] \cdot \frac{1}{F(x)} \cdot$$

This is a weighted average of η_0 and η_1, the weights being the quantities consumed at any given value of x. The proof holds for any number of individual curves.

identical tastes or systems of preferences, and (3) that the two incomes which will make them equally "well off" in the two situations under comparison must be known in advance. Now, it is evident that, as long as the theory rests on these assumptions, it will be impossible ever to compute a correct index number. Indeed, to be entirely free from theoretical objection, the whole procedure would have to be carried out in a "timeless" manner, by comparing hypothetically different price situations for only one individual. The situation would, therefore, appear to be utterly hopeless. Fortunately, however, this is a type of case where the *indirect* method of testing the admissibility of the assumptions is applicable, though the procedure is somewhat roundabout. But it may be worth while to describe it briefly, by way of example.

Choosing a case where the second of the above assumptions is patently inadmissible, namely, the comparison of the prices in two different countries I and O, let us suppose the results of Paasche's and Laspeyres' index formulae to be available for a whole series of incomes in each country. That is to say, let the q_0's and the q_1's in the two formulae: $\Sigma p_1 q_0 / \Sigma p_0 q_0 = P_0$ (Laspeyres) and $\Sigma p_1 q_1 / \Sigma p_0 q_1 = P_1$ (Paasche) be the quantities consumed by people having different incomes,[13] the prices p_1 and p_0 being the same for all income classes in each country. Then, according to Bortkiewicz,[14] we may write for the relative difference between any two values of P_1 and P_0

$$D = \frac{P_1 - P_0}{P_0} = r_{pq} \cdot \frac{\sigma_p}{P_0} \cdot \frac{\sigma_q}{Q_0}$$

where r_{pq} is the weighted coefficient of correlation between p_1/p_0 and q_1/q_0 and σ_p, σ_q are the weighted standard deviations of the p_1/p_0's and the q_1/q_0's respectively, the system of weights being the p_0q_0's throughout. Considering now that σ_p/P_0 and σ_q/Q_0 may be looked upon as measures of the dissimilarity of the two price and quantity systems involved, certain criteria may be developed which make it possible to judge the varying degree of "comparability" of the prices, according as they are weighted by coefficients into which the quantities consumed on different income levels enter. It must be noted that these criteria are established with a view of determining those combinations of r_{pq}, σ_p/P_0 and σ_q/Q_0 which, though leaving D always relatively small, seem least incompatible with the postulate that the tastes and habits must be as similar as possible between the two classes of

[13] In practice, these computations can only be carried out for foodstuffs, because the necessary data on quantities and prices are not available for the other sections of the budget.

[14] L. v. Bortkiewicz, "Die Kaufkraft des Geldes und ihre Messung," in *Nordic Statistical Journal*, IV (1932), 1–68.

people whose quantities are used as q_0's and q_1's in the computation.[15] Applying the criteria in order to choose among the various combinations the pair or pairs of P_0 and P_1 for which comparability is best, it will be found that the one or more combinations chosen are those where the value of σ_q/Q_0 is smallest. This has a particular significance. It can in fact be shown that the values of σ_q/Q_0, when they are calculated for a series of incomes in country I with respect to any given income in country O, generally reach a minimum for a certain income in country I, and that this minimum shifts when the calculations are repeated with respect to another income in country O.[16] When the values of σ_q/Q_0 are computed systematically for all possible income combinations between the two countries, then the surface obtained by putting in a three-dimensional diagram each value of σ_q/Q_0 against the two incomes which supplied the particular q_0's and q_1's used in its calculation will show a fairly definite and continuous "valley." Assume now (a) that all individuals in the same country have identical tastes and habits of consumption and live under identical circumstances except for the difference between their incomes, so that the dissimilarity (measured by σ_q/Q_0) between the quantities consumed in different income classes is due only to the differences in the (real) incomes, i.e., that they would all consume the same quantities, if they had equal incomes,[17] and (b) that the dissimilarity between individuals belonging to two different countries consists of an irreducible minimum, to be accounted for by the difference in the circumstances other than the real incomes (particularly the systems of wants and the price systems in the two countries), and a variable portion due to the difference in the real incomes. Then it can be concluded that the difference in satisfaction connected with the money-incomes between which the dissimilarity value σ_q/Q_0 is smallest has been reduced to a minimum, i.e., that the two incomes are more nearly equivalent to each other than either one of them is

[15] The criteria thus do not exclude the possibility that the tastes and habits of the people in one country may be different.

[16] This statement is based on a large number of experiments where use has been made, not of σ_q/Q_0, but of the formula

$$D' = \sum \left| \frac{p_0 q_1}{\sum p_0 q_1} - \frac{p_0 q_0}{\sum p_0 q_0} \right|.$$

Though the absolute values of this formula are different in size from the σ_q/Q_0's obtained from the same material, the movements of the two are closely similar, when the q_1's entering into their computation relate successively to different income classes in country I. See the present writer's article in ECONOMETRICA, II, (1934) 59–72.

[17] This evidently amounts to saying that every individual maximizes satisfaction on an identical system of wants.

to any other income. In other words, the application of the comparability criteria (which are based on assumptions (1) and (2) above) leads to the choice of one or possibly a certain number of combinations of incomes which satisfy condition (3).[18]

It is submitted that, whenever such roundabout methods lead to similarly self-consistent results, in spite of the fact that some of the secondary assumptions may be questionable, the main assumptions involved may be taken to be fulfilled nearly enough by the materials, even though no attempt has been made to verify this independently and directly. Since, however, such roundabout methods will not always be readily available, it is highly desirable, in view of the nature of budget materials, that econometricians should make attempts at obtaining such tests.

III. FAMILY BUDGET ENQUIRIES IN THE VARIOUS COUNTRIES

There is a third kind of difficulty which seriously hampers, not only the work of the econometrician, but also the solution of many practical questions: namely the scarcity of the family budget materials. To mention only a few examples, no post-war enquiries exist for Great Britain and France, none at all for Italy and Spain, while the most recent nation-wide enquiry for the United States was conducted in 1918–19. Although the econometrician may obviate this difficulty by using older enquiries, or enquiries relating to any country or countries, and trying to overcome their particular deficiencies from case to case, it would both serve him and surmount a serious obstacle to the study and solution of urgent, particularly social, problems, if a plan for an international enquiry could be carried out somewhat on the following lines:

1. Simultaneous and comparable family budget enquiries should be called forth in the main industrial centres of a series of important countries. These enquiries would, most probably, have to be carried out by the national statistical authorities of each country, after a conference of their competent representatives has agreed on the amount and kind of information to be provided. Reasonable income ranges would have to be covered in each town; comparable, if not identical, classifications for all the expenditure items should be adopted throughout; possibly identical scales for computing "equivalent adult males" would have to be agreed upon; the number and kind of families (as to location, social class, size and occupations) would have to be fixed; the period and duration of the enquiries should be chosen; the method of

[18] For a more detailed statement of the reasoning summarized above, see the author's "The International Comparison of Food Costs," to be published shortly in *International Comparisons of Costs of Living*, above quoted.

description to be used for housing, etc., should be the same throughout.

2. Provisions should be made for the periodical collection, in all the towns included, of retail price material of such a kind as to correspond, as closely as possible, to the data on quantities collected in the budget enquiries. It may be assumed that, if such price data could be secured, the budget material once collected might for certain purposes be used for at least five further years with a sufficient degree of approximation.

3. The value of the enquiry would be enhanced, especially for practical ends, if provision were made for the periodical collection also of wage material that would not only completely represent actual earnings (by occupations and kind of enterprise) but also be strictly comparable in time. These data would be valuable in themselves as showing the movement of income in important classes during time; beyond this, they would offer, by a possible connection with the family budget enquiries, an ideal material for interlocal and international comparisons of real wages.

The information on budgets and prices collected would make possible the construction of comparable indices showing the variations in the cost of living in the course of time for each town and for different levels of income; this would seem a considerable improvement upon existing index numbers of this kind which are not only constructed quite differently, and are therefore quite non-comparable, in different countries, but also do not generally distinguish between either different localities in each country, or different levels of income.

Further, indices could be constructed which would permit of comparing employees' incomes in any two or more countries or towns. Reliable local, interlocal, and international measures would thus become available both for judging certain controversial aspects of social questions, and, if such a course is desired, for showing the possible need, and supplying the basis, for any kind of international wage policy, so far as standards of living are to play any part in it.

International Labour Office, Geneva.

(*A table which may serve as a guide to the more important family-budget-materials that are at present available will appear in the January issue of* ECONOMETRICA.)

ON EQUATIONS OF MOTION OF BUSINESS ACTIVITY

By F. Creedy

SUMMARY

The present investigation aims at basing the subject of Economic Dynamics on clear mathematical foundations as rigorous as those employed in any other branch of dynamics. It is shown that it may be based on postulates in complete formal analogy to those of ordinary dynamics. Economic Inertia and Economic Resilience (and Storage) are then defined and illustrated by examples. Differential equations involving these are next formulated for simple cases corresponding to the ordinary Dynamics of a Particle and it is shown how they enable us to plot curves of economic behavior as functions of time. Some of these curves are found to be oscillatory and others not. In Part II an endeavor is made to state the problem of economic dynamics in a sufficiently general form to permit of the immediate application of the modern methods of dynamical analysis used in physics. It is pointed out that, just as in radiation, we have periodic phenomena in economics, since most of our statistics show economic cycles. Our problem is essentially the same as the physicist's since both may be stated in the same words: "Given a jumble of periodic phenomena, to find an interconnected dynamical system which will parallel the observed phenomena without departing at any point from what we can observe in other manners." An example is worked out illustrating our general dynamical system in a simple way, and the investigation closes with some general observations.

THE NOTION OF ECONOMIC FORCE

If a salesman calls at my house to persuade me to use something I did not use before—for instance breakfast food A instead of B—the measure of his success is the *rate* at which he can change my consumption. If he can persuade me entirely to discontinue B and substitute A as soon as present stocks are used up, he has exercised very powerful persuasion. If he can only persuade me to give A a trial, and by repeated visits to increase my use of A, his persuasion is less powerful. If we wish to have a numerical measure of the effective persuasive force which he exerts at any time, it is clear that our best measure is in terms of the rate of change in the consumption per unit time he brings about. This may be conveniently called the *acceleration* of consumption. Persuasion may, of course, be exercised for other purposes

than the increase of consumption. More generally therefore we may define

Effective Persuasive Force = Rate of Acceleration of Economic Action times a Constant.

Economic Action is itself a rate. It is measured, for instance, by production per day or consumption per day (or other units may be used). These may be called velocity of production (or of consumption). In mathematical language, if X is the amount of goods produced from an arbitrary time (the beginning of the year or any other moment) up to now (which we refer to as time t) then

$$\dot{X} = \text{Velocity of production} = \frac{dX}{dt} \text{ (tons per day, say,)}$$

$$\ddot{X} = \text{acceleration of production} = \frac{d^2X}{dt^2} \text{ (tons per day, per day).}$$

This means that if we produced 10 tons yesterday and 12 tons today, the acceleration of production has been 2 tons per day in one day.

If F = persuasive force and M is a constant so far undefined, our definition in mathematical language will read

$$F = M\frac{d^2X}{dt^2},$$

and as soon as we know M (which is connected with the so-called "sales resistance", which we prefer to call the "sales mass," of different individuals) we can calculate F.

This is clearly identical in form with Newton's third law

$$F = M\frac{d^2X}{dt^2}$$

Force = Mass × Acceleration.

Just as he gave precision and measurability to the tactile and muscular sensations connected with pushing and pulling by this definition, so we endeavor by a parallel method to give precision and measurability to the psychological ideas connected with different degrees of persuasion.

Of course, in writing down such equations, we postulate that physical conditions must remain the same. Acceleration of production from a mine might merely mean that a richer lode or seam had been struck, and not that the workers had been persuaded to increase their effort.

The first law of motion.—In place of Newton's first law, we have the

statement "If there is no persuasive force tending to make us change our minds, we shall go on indefinitely producing and consuming goods and services at the same rate." Economic affairs will have arrived at a steady state. This "law" is surely obvious.

The second law of motion.—Our second law is also of a very simple nature. Incomings must balance outgoings—in money where money is used, otherwise in some other customary unit of measurement satisfactory to the parties. More briefly "The books must balance." This principle enables us to set up equations among our quantities.

The third law of motion.—Recapitulating our third "law," or rather definition, it is

Effective Persuasive Force = Rate of Acceleration of Economic Action times a Constant.

Economic action may be production, consumption, or various other things. If the Persuasive Force is devoted to production, then it is measured by the acceleration of production, and so on.

Thus we have a set of principles, in complete formal analogy with Newton's laws, which will enable us to state and solve differential equations of motion for economic quantities so that these quantities may be determined as functions of time.[1]

THE PRINCIPLE OF ECONOMIC INERTIA

Just as Newton's first and third law implicitly contained the principle of mechanical inertia, so our corresponding relations contain that of economic inertia. Both these principles may be stated as follows: "Velocities (of production, consumption etc., or of masses) can only change at a finite rate." This is just as obvious in the economic as the mechanical case.

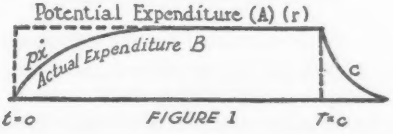

FIGURE 1

Some instances of economic inertia are the following.

1. The Munitions industry.—At the beginning of the war an enormous demand for munitions developed but nevertheless it could only gradually be satisfied. If the demand is roughly represented by the amount of money per day the Legislature was willing to (but could

[1] F. Creedy, *The Secret of Steady Employment* (G. P. Putnam's Sons, 1933), p. 100.

not) spend (suppose constant) as at A, Figure 1, then the supply (or actual expenditure) will be represented by B, only gradually rising to equal A.

The demand for munitions was absolutely cut off by the armistice, yet it was impossible to stop manufacture and supply instantly. The decrease is shown in curve C. The writer witnessed the testing of electrical machinery for the British Munitions Ministry in the spring of 1919. Contracts could not be cancelled abruptly nor enormous masses of men and women thrown abruptly out of work; there was a host of considerations such as these.

2. Exactly the same type of curve will be found if we regard A as the income which an individual suddenly gets (for instance by his obtaining a new post) at $t=0$. It will take him time to get accustomed to his new scale of living and he will only be able gradually to increase his expenditure more or less in accordance with curve B. "I have $10,000 available. Well, what shall I buy." It will take a certain period of time, if only five minutes, to answer this question. If at $t=c$ he loses his post, so that his income is reduced again, he cannot immediately reduce his expenditure, but it must take a finite time (no matter how short) to taper off. That is, it will approximately follow curve C.

This problem of the rate at which a man spends his income is one of the simplest in economic dynamics yet nevertheless will enable us to illustrate many features of our general method.

Each distinct type of problem contains some particular psychological hypothesis as to the relation of our Persuasive Force to other quantities. This must be arrived at by observation, in the same way in which the law of falling bodies was arrived at, for instance. For the purpose of our present example we shall assume:

Constant times Rate of acceleration of spending
= Unspent balance of Income.

This means that if the individual has a fixed amount of money to spend in a given period, he will spend at a faster rate and with less circumspection when he has a large unspent balance than when he has a small one. An illustration of this is the appropriation for departmental expenses in a university. When the appropriation has just become available at the beginning of the session, approved projects requiring expenditures are put in hand fairly rapidly. At the end of the session when the appropriation is largely used up it is much more difficult to put in hand new projects, however desirable. Many other instances similar to this will occur to anyone. When a man or an organized group has a large unspent income he "feels flush" and is

ready to spend. When he has little unspent income he is very careful how he spends what is left.

While this may prove that the acceleration of spending is greater when the unspent balance is large than when it is small, it does not prove, of course, that they are directly proportional. This latter assumption we justify on the ground of simplicity. Since there clearly is a relation between unspent balance and acceleration of spending and we are quite vague as to its exact nature while anxious not to ignore it, it is better to assume that, to a first approximation, it is linear and see whether this gives results reasonably in accord with observation. If not, we can perhaps correct it.

In mathematical language if s is unspent balance we may say generally

$$\text{Acceleration} = F(s)$$

$$= F(0) + sF'(0) + \frac{s^2}{1.2}F''(0) + \cdots$$

by Maclaurin's theorem.

If we assume that the curve of $F(s)$ is so smooth that $F''(0)$ and succeeding coefficients can be neglected, we have a linear relation. We also assume that $F(0) = 0$. This is obvious. It means "No acceleration of spending if there is no money to spend."

Let us now set up a balance-sheet (in accordance with the second law) for the individual under study.

Let P = average price of goods & services purchased (a constant)

$\dot{x}$ = volume of goods purchased per unit time so taken that

$P\dot{x}$ = total expenditure

$L d\dot{x}/dt$ = constant times Acceleration of spending = Unspent balance.

r = Income per unit time.

Balance Sheet

Cr.			Dr.
By income	r	To expenditure $P\dot{x}$	
		To unspent balance $L\dfrac{d\dot{x}}{dt}$	

Equating the two sides

$$r = P\dot{x} + L\frac{d\dot{x}}{dt}.$$

This is an equation of the first order in x with constant coefficients such as is of constant occurrence in many branches of science. It is not necessary therefore to go through the mathematical processes of solution so we can merely state that the solution is

$$\dot{x} = \frac{r}{P} + \left(\dot{x}_0 - \frac{r}{P}\right) e^{-Pt/L}$$

where $\dot{x}_0$ is the amount being purchased per unit time when $t=0$. If $r=0$ the solution is

$$\dot{x} = \dot{x}_0 e^{-Pt/L}.$$

If we plot these equations we get curves of exactly the shape of those of Figure 1. If we suppose Figure 1 the result of observation, then by a suitable choice of r, L, and P, we may obtain a maximum closeness of fit between the experimental and calculated curves. Such a test, if successful, would be an argument in favor of studying the spending of a man's income in the way we have done. Thus, we first of all frame our psychological hypothesis after careful study of the actual problem, then deduce the consequences and finally note whether these consequences are in agreement with observation. If not, we modify our hypotheses, gradually getting nearer and nearer to observation by successive modifications. This is, of course, the universal method of scientific reasoning, but on account of the queer objections that are from time to time formulated against the use of mathematics in economics it is as well to show how the method of successive approximations appears in our present subject.

Returning to Figure 1, expenditure $P\dot{x}$ starts from zero at $t=0$ and gradually rises to the final value of r where it stays indefinitely. If r is cut off, expenditure $P\dot{x}$ rapidly, but not instantly, falls off in accordance with the curve

$$P\dot{x} = P\dot{x}_0 e^{-Pt/L}$$

the unspent balance being negative (borrowings).

THE PRINCIPLE OF ECONOMIC RESILIENCE[2] OR STORAGE

So far, in considering the expenditure of an individual, we have assumed that what he did not spend he did not have, a principle sometimes made use of in allowances for various purposes. He was unable to put his unspent balance in the bank and draw interest on it.

If, starting at time T, a payment r per unit time is made into a bank, then the total capital accumulated (without interest) up to time t will be

$$\int_T^t r\,dt$$

[2] The term "elasticity" is employed by economists in a totally different sense, so it is here avoided.

and the total (not compounded) interest at rate C will be

$$C\int_{T}^{t} r\,dt.$$

If the time T is arbitrary, the lower limit merely gives us the arbitrary constant and we may use the indefinite integral writing

$$\text{Interest} = C\int r\,dt.$$

Let us restate the balance sheet of our last example on the supposition that the unspent balance is in the bank receiving interest at rate C.

Balance Sheet at time t

Cr.		Dr.
By income r	To expenditure $P\dot{x}$	
By interest on Unspent Balance $C\int(r - P\dot{x})dt$	To unspent Balance $L\frac{d\dot{x}}{dt}$	

$$r + C\int (r - P\dot{x})dt = P\dot{x} + L\frac{d\dot{x}}{dt}.$$

We assume that r is constant.

Differentiating to get rid of the integral sign we get

$$C(r - P\dot{x}) = P\frac{d\dot{x}}{dt} + L\frac{d^2\dot{x}}{dt^2}$$

that is,

$$r = P\dot{x} + \frac{P}{C}\cdot\frac{d\dot{x}}{dt} + \frac{L}{C}\cdot\frac{d^2\dot{x}}{dt^2}.$$

This again is an equation of a kind that pervades all science so that very little of the detailed processes of solution need be given.

If we temporarily write d/dt as D, the equation may be written

$$\left(\frac{L}{C}D^2 + \frac{P}{C}D + P\right)\cdot\dot{x} = r$$

and the solution may be of two distinct forms according to the values of the constants.

If we had $\left(\frac{L}{C}D^2 + \frac{P}{C}D + P = 0\right)$

a quadratic equation, this would give

$$D = \frac{-\frac{P}{C} \pm \sqrt{\frac{P^2}{C^2} - 4\frac{LP}{C}}}{2\frac{L}{C}}$$

$$= -\frac{P}{2L} \pm \sqrt{\frac{P^2}{4L^2} - \frac{CP}{L}}.$$

If $CP/L > P^2/4L^2$ the square root or "discriminant" will be imaginary and the solution of the equation will be of the form

$$\dot{x} = \frac{r}{P} + Ae^{-Pt/2L}\cos(wt + a)$$

where A and a are arbitrary constants and

$$w = \sqrt{\frac{CP}{L} - \frac{P^2}{4L^2}}.$$

If however $CP/L < P^2/4L^2$ the square root will be real and the solution will be of the form

$$\dot{x} = \frac{r}{P} + Ae^{(-P/2L+\sqrt{P^2/4L^2-CP/L})t} + Be^{(-P/2L-\sqrt{P^2/4L^2-CP/L})t}$$

where A and B are constant. Both of these expressions are exponentials which decrease with increasing time though at different rates.

The first fundamentally important point we notice is that if

$$C > \frac{P}{4L} \quad \text{or, in other words,}$$

$$\frac{C}{P} > \frac{1}{4L}$$

the solution becomes oscillatory.

If the rate of interest in proportion to prices is too high, oscillations may occur.

If L is large a small rate of interest may set up oscillations.

A small rate of interest may be capable of setting up oscillations if there is great economic inertia.

If the solutions of these equations for the non-oscillatory and os-

cillatory cases are completed and plotted against time, they will appear as in Figure 2 and Figure 3.

In Figure 2 the cross-hatched area represents unspent income which accumulates in the bank drawing interest. It therefore permits spending later on (shaded area) which gradually reduces to the steady income.

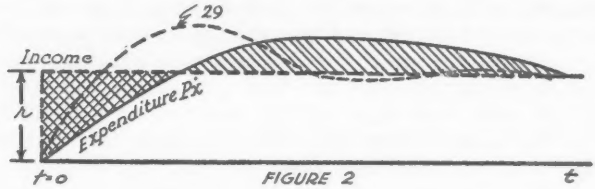

FIGURE 2

Figure 3 presents cycles of borrowing and repayment. During the cycle in which the individual exceeds his income he borrows from the bank, later on reducing his expenditure below his income in order to repay. In the first half cycle he also spends the money saved during the initial period (cross-hatched). He then saves during the second

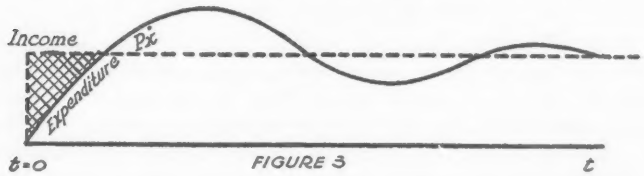

FIGURE 3

half cycle enough to repay borrowings in the first period and somewhat more which he proceeds to spend with some fresh borrowing during the third half cycle and so on. These oscillations, however, rapidly die down till finally he is only expending his steady income. In actual fact curves like Figure 2 in which all beyond the second half wave is suppressed by the heavy damping are more likely to occur than those of Figure 3. Although we are not in the habit of plotting our expenditure in such curves as Figure 3, most of us have similar cycles in our expenditure, during part of which we borrow from the bank, and later on economize to pay back. But of course such cycles are seldom regular and not always decreasing like the cycles of Figure 3.

From the point now reached, the meaning of Economic Resilience should be quite clear. Economic Resilience permits us at one time to

deposit surplus money in the bank on which the banker pays us interest. This gives him a strong motive for lending equivalent money so as to receive as much as he is compelled to give out. At another moment we may borrow money from the bank, paying interest on it proportional to the amount borrowed, thus giving us a strong motive for paying it back as soon as possible. Interest thus appears as a disciplinary measure whose object is to compel repayment of the principal.

Thus, just as we may store energy in a spring or electricity in a condenser, so we may store money in a bank. Just as the storage of energy in a spring produces a force tending to cause a return to equilibrium, so the storage of money in a bank produces a requirement for interest payments which again tends to cause a return to equilibrium.

Just as the forces of action and reaction on a body must sum up to zero (by Newton's second law) so the incomings and outgoings of a given individual or corporate body must also sum up to zero by the principle, "The books must balance."

The cycles we have considered are cycles of borrowing and repayment. Without banks and bank interest such cycles would be impossible. There might be masses in our economic system but there would be no springs, and hence no oscillations. It is worth considering whether the economic cycle in general is not of the same nature.

SOURCES AND SINKS OF CURRENCY

If we put money into a bank and then take it out later, we can take more out than we put in. Again, new money is created by every bank loan and destroyed when it is paid back. Hence, except in virtue of special conditions, a bank is a *Source* whence new currency may flow, or a *Sink* in which it may disappear. Thus our dynamical systems may contain Sources and Sinks (to adopt physical phraseology). Other types of source and sink may exist, for instance the Treasury Department. Its power of note issue is a source, and securities sold at high prices which afterwards become worthless, are sinks.

INTERCONNECTIONS BETWEEN INDIVIDUALS

The examples just given in which all the relevant facts can be stated in a single balance sheet, correspond to the ordinary dynamics of a Particle. In this subject, the unbalanced resultant of all forces accelerates the particle, while in our case the resultant of all incomings and outgoings or unspent balance accelerates the operation under study. This will be clearer if we develop a diagram showing the lines of money flow.

It will be simplest for our purpose to assume a purely cheque currency. Under this assumption, whenever a payment is made money

flows out of the bank, effects the payment, and returns to the bank again. If a continuous series of payments of the same kind (for instance purchases of goods or payments of wages) is being made, there is a continuous flow of money out of the bank and back again.

Denoting the individual whose manner of spending his income we are studying (our "particle") by a small circle, the different types of incoming or outgoing may be represented by different lines of money flow as in Figure 4, the resultant of these, or "unspent balance," being devoted to accelerating his consumption.

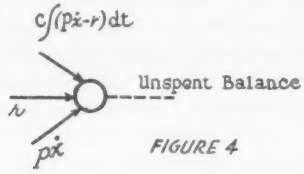

FIGURE 4

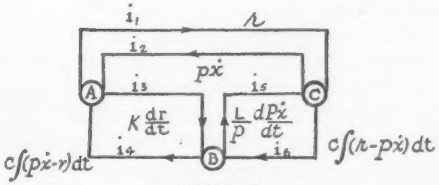

FIGURE 5

Convention.—In what follows one line of money flow to our individual (who interests us only as a re-distributing point for money flow) will be devoted to income r, another to purchases $P\dot{x}$, another to interest $C\int(P\dot{x}-r)dt$, another to unspent balance, and others as required to other types of expenditure and income, all per unit time. This convention permits us to represent the more complicated cases we wish to study by diagrams which serve to bring the problem vividly before the mind. Let us draw such a diagram for a case where three persons are concerned, or, in other words, there are three distributing points for money flow.

Let us suppose that A is the producer of goods $\dot{x}$ per unit time and that he employs C at wages r to assist in production, C being likewise the consumer purchasing the goods $\dot{x}$ at price P. Hence A has the amount $P\dot{x}$ flowing in from the proceeds of sales. The difference between $P\dot{x}$ and r he puts in the bank B drawing interest $C\int(P\dot{x}-r)dt$. Similarly C puts the difference between his incomings and outgoings

in the bank drawing interest, $C\int(r-P\dot{x})dt$. Let i_r be the money flow in the rth channel. By our second law

(1) $i_1+i_2+i_3+i_4=0$ taken as outflowing from A.
(2) $-i_3-i_4-i_5-i_6=0$ taken as outflowing from B.
(3) $-i_1-i_2+i_5+i_6=0$ taken as outflowing from C.

If Money i_1 flows out of A it flows into C. This is equivalent to saying that $-i_1$ flows out from C. This accounts for the signs. We have six different money flows and only three equations. We require three more equations to enable us to obtain solutions. These further equations may be called equations of connection.

Equation (2) above requires a little further elucidation. In the present example the bank B is merely a distributing point and never a store of currency. Therefore, if money is being paid into one account, an equal amount is being drawn out of another. If interest is being paid to one depositor, an equal amount of interest is being paid by another depositor. If these assumptions are not made, the bank could be a Source or Sink of currency, that is a storehouse, which is contrary to our initial assumption.

Let i_3 be the unspent balance paid into the bank by A.

Let i_5 be the unspent balance paid into the bank by C. Now i_3+interest $=P\dot{x}-r$ to satisfy eq. (1), but the banker has no simple mechanism to enable him to tell the amount i_3 which with its interest $-C\int i_3dt$ subtracted will amount to $P\dot{x}-r$. In fact, to do this would require the solution of the differential equation,

$$i_3 - C\int i_3dt = P\dot{x} - r \text{ or } \frac{di_3}{dt} - C\cdot i_3 = \frac{d(P\dot{x}-r)}{dt}.$$

(Interest is negative because it flows in the opposite direction to principal.)

So he will simply allow interest on the smaller amount $P\dot{x}-r$. At any rate, that is what we assume in the present example. Thus we have

$$i_4 = -C\int (P\dot{x}-r)dt \tag{4a}$$

$$i_6 = -C\int (r-P\dot{x})dt. \tag{4b}$$

Since $i_4+i_6=0$, it follows from (2) that

$$i_3 + i_5 = 0. \tag{4c}$$

The equations (4abc) give us only one further condition since they both flow from the assumption: Interest received at any distributing

point is to be reckoned on the unspent balance at that distributing point. From this principle and the previous equations both (4a) and (4b) flow hence they are not independent and give us only one new relation as stated.

One of the further equations of connection will be the psychological hypothesis made in connection with the previous examples, viz.,

$$i_5 = \text{Unspent balance} = L\frac{d\dot{x}}{dt} \text{ or } \frac{L}{P}\cdot\frac{d(P\dot{x})}{dt}. \tag{5}$$

Throughout we shall assume $P=$constant or $dP/dt=0$.

To give us our final equation of connection and make the example capable of definite solution we require another psychological hypothesis. We shall assume: When there is an unspent balance of profits (i.e., $(P\dot{x}-r)$ is positive) wages r will show a rising tendency (perhaps very slight) and when $P\dot{x}-r$ is negative (losses exist) wages r will show a falling tendency. Again, we assume a linear relation merely for simplicity. This assumption yields the equation

$$P\dot{x} - r = k\cdot\frac{dr}{dt}.$$

The same equation may be derived by widely different arguments. It has been pointed out[3] that wages at any instant can only be paid out of surpluses existing at the previous instant. The surplus existing at time t is $\int(P\dot{x}-r)dt$, since $P\dot{x}-r$ is the unspent balance of profits and the summation of this is the total surplus available at any instant. If r is proportional to this we have

$$\int (P\dot{x} - r)dt = k'r$$

or, differentiating,

$$P\dot{x} - r = k'\frac{dr}{dt} \tag{6}$$

(k & k' derived by these two arguments need not be equal). It should be noted that while the first derivation involves a psychological assumption, the second does not, and leads also to the presumption that $k=1$ in many cases. This gives us our first illustration of the fact that the same equations may be derived by many different arguments.

[3] Keynes, *Treatise on Money*, Harcourt, Brace and Company, New York, 1930, Vol. II.

Recapitulating, our equations are (1), (2), (3), (4abc), (5), and (6), where only one of the three equations (4abc) is independent.

If we write $i_1=r$, $i_2=-P\dot{x}$, this is a mere change of nomenclature and does not give us any further equations.

We now have

$$r - P\dot{x} + k\frac{dr}{dt} - C\int (P\dot{x} - r)dt = 0 \tag{7}$$

$$P\dot{x} - r + \frac{L}{P}\cdot\frac{dP\dot{x}}{dt} - C\int (r - P\dot{x})dt = 0 \tag{8}$$

$$\text{with} \qquad k\frac{dr}{dt} + \frac{L}{P}\cdot\frac{d(P\dot{x})}{dt} = 0. \tag{9}$$

Two of these equations are independent, and solutions for $P\dot{x}$ and r may therefore be obtained.

The method adopted above is given so as to show clearly the assumptions involved and so as to be capable of generalization. The connections may also be exhibited very simply by means of the balance sheets.

Producer (A)

By sales $P\dot{x}$	To wages r
By interest $C\int(P\dot{x} - r)dt$	To unspent balance $K\frac{dr}{dt}$

Consumer (C)

By income r	To purchases $P\dot{x}$
By interest $C\int(r - P\dot{x})dt$	To unspent balance $\frac{L}{P}\cdot\frac{d(P\dot{x})}{dt}$

Bank B

By receipts from producer $-k\frac{dr}{dt}$	To interest paid producer $C\int(P\dot{x} - r)dt$
By receipts from consumer $-\frac{L}{P}\frac{d(P\dot{x})}{dt}$	To interest paid consumer $C\int(r - P\dot{x})dt$

This gives directly the equations (7), (8) and (9). Equation (9) may be immediately integrated giving

$$kr = -\frac{L}{P}\cdot P\dot{x} - C_0 \text{ or } kr + C_0 = -\frac{L}{P}P\dot{x} \tag{10}$$

where C_0 is an arbitrary constant.
Subtracting (7) and (8) we get

$$\begin{aligned} 2(P\dot{x} - r) + 2C\int (P\dot{x} - r)dt &= \frac{k\cdot dr}{dt} - \frac{L}{P}\frac{d(P\dot{x})}{dt} \\ &= -2\frac{L}{P}\frac{d(P\dot{x})}{dt} \text{ from (9).} \end{aligned} \tag{11a}$$

Substituting we get

$$2\left(P\dot{x} + \frac{L}{kP}P\dot{x} + \frac{C_0}{k}\right) + 2C\int\left(Px + \frac{L}{kP}P\dot{x} + \frac{C_0}{k}\right)dt = -2\frac{L}{P}\frac{d(P\dot{x})}{dt}$$

$$2\left(1 + \frac{L}{kP}\right)P\dot{x} + 2C\int\left(1 + \frac{L}{kP}\right)P\dot{x}\,dt + 2\frac{L}{P}\frac{d(P\dot{x})}{dt}$$

$$= -\frac{2C_0}{k} - 2C\int\frac{C_0}{k}dt$$

$$2\left(1 + \frac{L}{kP}\right)\frac{d(P\dot{x})}{dt} + \frac{2C}{k}\left(1 + \frac{L}{kP}\right)P\dot{x} + 2\frac{L}{kP}\frac{d^2(P\dot{x})}{dt^2}$$

$$= -\frac{2CC_0}{k}$$

which gives the symbolic equation

$$2\frac{L}{P}D^2 + 2\left(1 + \frac{L}{kP}\right)D + 2C\left(1 + \frac{L}{kP}\right)(P\dot{x}) = -\frac{2CC_0}{k}. \tag{12}$$

hence

$$D = \frac{-2\left(1 + \frac{L}{kP}\right)}{4\frac{L}{P}}\left(1 \pm \sqrt{1 - 4\frac{C\cdot Lk}{L+kP}}\right)$$

$$= -\frac{kP+L}{2kL}\left(1 \pm \sqrt{1 - 4\frac{C\cdot Lk}{L+kP}}\right). \tag{13}$$

This can be oscillatory for a sufficiently large value of C.

The steady state solution is $P\dot{x} = -C_0P/L+kP$ and $P\dot{x}$ approximates to this as t advances.

In the equation (12) we may substitute from (10). Collecting up we again get an equation whose solution may be oscillatory.

$$2\frac{dr}{dt} + 2Cr + \frac{2Lk}{L+kP}\cdot\frac{d^2r}{dt^2} = -\frac{2CC_0 \quad P}{L+kP}.$$

In symbolic form

$$\left(\frac{2Lk}{L+kP}D^2 + 2D + 2C\right)r = \frac{2CC_0P}{L+kP}$$

which gives

$$D = -\frac{L+Pk}{2Lk}\left(1 \pm \sqrt{1 - \frac{4LCk}{L+kP}}\right) \tag{14}$$

Thus in the oscillatory case the period and the rate of decrement are the same as before and the two quantities $P\dot{x}$ and r can differ only by their terminal conditions.

The steady state solution for r is

$$r = -\frac{C_0 \quad P}{L+Pk}.$$

Substituting for r and $P\dot{x}$, in equation (10), it is easy to check that (10) is satisfied by the solutions given.

If $P\dot{x}=0$ when $t=0$, the explicit form of the solutions are

$$kr = -\frac{C_0Pk}{L+Pk}(1 + \frac{L}{Pk}e^{-bt}\cos wt)$$

$$P\dot{x} = \frac{C_0P}{L+Pk}(e^{-bt}\cos wt - 1).$$

Take as an example $k=1$, $-C_0=20$, $L=1$, $P=1$. If we further take $C=.6$ (only high rates of interest produce oscillations in this example) we get the two curves shown in Figure 6.

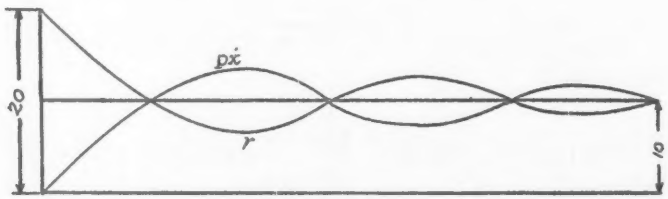

FIGURE 6

When $P\dot{x}$ is greater than r, the wage earner is spending more than he is earning but borrowing the surplus from the profit-paid worker, since

profits ($P\dot{x}-r$) are positive. When $P\dot{x}$ is less than r the wage-earners are spending less than they earn and lending the balance (doubtless through the money market) to the entrepreneur who is spending more in wages than he is taking in from sales.

THE GENERAL PROBLEM OF ECONOMIC DYNAMICS STATED

We are now in a position to state at least one of the general problems of our embryo dynamical science. We shall define a "dynamical system in economics" as a system of r monetary distributing points including producers, consumers, banks (which may be sources or sinks of money flow), etc., producing a network of flows, as shown in Figure 7.

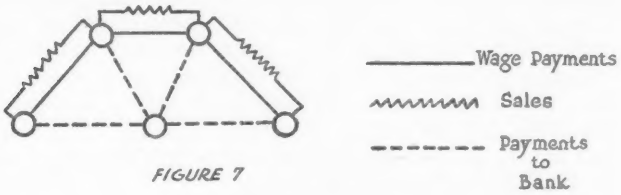

FIGURE 7

We set up a balance sheet for each distributing point. Each balance sheet will, as a rule, contain entries relating to other balance sheets, which establishes a relation between them. These relations must be expressed as functions of the variables of the system which are the monetary flows along the different paths and their derivatives. By convention, we establish a different path for each kind of flow, one for the purchase of goods, for instance, one for interest, and one for the flow of unspent balances to the bank, another for wage payments. These may be shown by different kinds of line as in Figure 7. Other paths may be set up for other kinds of flow. The relations between corresponding entries in the balance sheets must be expressed by "equations of connection" shortly called "connections." These equations must be sufficient in number to make the total number of variables equal to that of the equations. Among these will be found the psychological hypotheses referred to above and also equations resulting from social institutions such as banking. It will be evident to the student of dynamics that the Lagrangian equations can be applied to such a system, but their detailed deduction is reserved for another occasion.

Having formulated our equations in some such way, the problem is: "Given certain unspent money flows, how in accordance with the connections of the system will they be spent?"

In practice, however, the problem that comes to us is the converse of this. Exactly as in atomic physics, we can only *observe* statistical averages and not individual occurrences. We have statistics of various kinds of production, of bank clearings, etc., and from these we have to infer the connections of the system, guided in our case by our knowledge of what they actually are in many individual cases, just as the physicist is guided by the result of individual experiments. If our hypotheses as to the internal connections of the system do not bring out results in accordance with the statistical evidence, we must modify and modify again till they do. Just as in radiation, we have periodic phenomena, since most of our statistics show an economic cycle. We have no such convenient instrument as the spectroscope (although mechanical harmonic analysers might serve the same purpose) to resolve our periodic phenomena into its component simple harmonic oscillations, but our problem is essentially the same. "Given a jumble of periodic phenomena, to find an interconnected dynamical system which will parallel the observed phenomena without departing at any point from what we can observe in other manners." This is a statement of the problem which is applicable without changing a word to either the physical or the economic case.

Modern theoretical and statistical dynamics has shown that a large number of useful theorems can be proved about dynamical systems, without knowing the exact equations of connection. Our first task therefore must be to adapt these generalised results to our form of dynamics so as to render our tools for research as effective as possible.

Lehigh University

THREE CONCEPTS OF WAGE QUOTAS AND THEIR STATISTICAL DETERMINATION

By O. Bakker

In periods of economic depression, when prices are falling and reach a level where manufacturing is working at a loss, the entrepreneur tries to lower his costs of production in order to bring them into accordance with the reduced selling-prices.

It goes almost without saying that a reduction in wages and salaries is considered, which, as a rule, meets with opposition from the side of the wage-earners and employees.

Among the arguments put forward in this discussion, we can regularly find the conflicting assertion, on the one hand (from the employers) that wages play an important part in the cost of production, and on the other hand (from the wage-earners and employees) that the influence of wages is so slight that a reduction can only have a small influence on the cost of production of products.

As an example of this last argument we quote here a statement of Eduard Weckerle[1] who, starting from figures about the U.S.A., U.S.S.R., Norway, Canada, and Germany, concludes (pages 73–74): "*die anteilmäszigen Lohnkosten liegen im Durchschnitt aller Industrien selbst in den Ländern der höchsten Löhne unter 20 Prozent.*"

Another example can be found in the paper: "Le prix de la main d'œuvre dans les prix de revient," published in *La voix du Peuple*[2] (Sept. 1933, 4th Série, No. 156). From the American Census figures this article concludes that wages, on the average, constitute only 17.4 per cent of the cost of production, so that a shortening of the working-week from 48 hours to 40 hours with unchanged weekly wages, would raise costs by only 3 per cent.[3]

Prof. Dr. Friedrich Zahn[4] comes to an opposite conclusion when he writes in *Soziale Praxis* (40. Jahrgang, Heft 23): ". . . *dasz der Lohn in der Volkswirtschaft weitaus das wichtigste Kostenelement ist, welches das volkswirtschaftliche Preisniveau oder die Kaufkraft des volkswirtschaftlichen Betriebskapitals ausschlaggebend, mindestens mit einem*

[1] *Die Krise des Kapitalismus und die Aufgabe der Arbeiterklasse*, Verlag der Marxistischen Büchergemeinde, Berlin: Brits.

[2] Organ of the French Federation of Trade-Unions.

[3] This should be, owing to a wrong calculation in the above-mentioned paper, 3.5 per cent.

[4] Professor Zahn refers to a statement of Adolf Weber. The latter, however, gives no further proof for his assertion (vide *Conrads Jahrbücher*, (Jena, 1931), CXXXIV, 707).

Gewicht zwischen 80 und 90 per cent der gesamten Gestehungskosten, bestimmt."

This conclusion of Professor Zahn was opposed in the same periodical (41. Jahrgang, Heft 17) by Dr. J. Marschak, who, *i.a.*, puts forward that, in Germany in 1925–29, wages and salaries constituted about 57 per cent of the national income and consequently a percentage between 80 and 90 per cent was not possible.

These controversies can only be solved by stating more precisely what is meant by the proportion which wages hold to total cost of production and then actually determining this proportion in a quantitative way on the basis of statistical information.

The problem of the relation between wages and cost of production can be put in three ways:

I. the relation between wages and the cost of production in the individual establishments: *the enterprise wage quota.*

II. the relation between wages and the cost of production of a certain commodity: *the commodity wage quota.*

III. the relation between wages and the total national production, whereby the whole national economy is to be considered as a single unit: *the national wage quota.*

I. The wages paid in a certain establishment is a notion that can be outlined rather sharply and, when the entrepreneurs are willing to cooperate, the cost of production and the wages paid in those separate establishments can be ascertained without great difficulties.

This wages' quota is meant by Weckerle and *La voix du Peuple*, and indeed, when starting from this conception of "the relation between wages and the cost of production," the handling of the available material may lead to the conclusion that the wages' quota amounts to about 20 or 25 per cent.

II. The definition of "the relation between wages and the cost of production of a certain commodity" presents more difficulties. It is obvious that the wages should comprise all wages paid in behalf of the commodity in the *successive phases of production and distribution.* For example, with regard to boots and shoes, not only the wages paid in the boot and shoe factory should be taken into account, but also the wages paid in the shoe store for the distribution of boots and shoes. In this way an "accumulated wages' quota" is obtained. But this accumulated wages' quota should include even more: in the boot and shoe factory the boots and shoes are made of leather and the price of this leather also contains wages, viz., the wages paid in the tannery for the manufacture of leather. The fuel and packing material used

in the leather and boot and shoe factories, the freight paid for the transportation of raw material and finished product, all these contain wages, and these wages should be added to the wages' quota in the price of the boots and shoes. Consequently the accumulated wages' quota should embrace all wages paid by trades and factories which have joined in the process of production and distribution, either directly or indirectly.

The percentage calculated by Professor Zahn, (see above; the commodity for which he made his calculations, is "Zierporzellan") is an example of such an accumulated wages' quota.

As to the inquiry in the Netherlands, the restriction was made that only those wages were to be considered which were earned in establishments located in Holland.

When in a struggle for higher (lower)wages between the employer and the employee the two parties play off against each other a different wages' quota, it is clear that the one party is using the wages' quota I and the other the wages' quota II. Though we do not wish to take sides in this controversy, it is evident that when we want to trace the influence of a general wage reduction on the cost of production, we should start from the latter method, the accumulated wages' quota.

III. For the application of this method we must know the national income and the part of it formed by wages. The wages' percentage of about 57 per cent, mentioned by Dr. Marschak (*vide* above), was ascertained in this way. (This method, illustrated with examples, will be discussed at the end of this article).[5]

At the beginning of 1931 the Dutch Government charged the Central Bureau of Statistics with an investigation into *the influence of wages and salaries on the costs of production*. The results were published in August 1933, and a short report is given below of the statistical data contained in this publication, whereby the above-mentioned controversies will come into discussion.

We shall first give a definition of the conceptions "wages" and "cost of production." In this inquiry the Netherlands Central Bureau of Statistics ranked among "wages": wages and salaries, including the entrepreneurs' labor income (if necessary estimated), commissions for travellers, and social charges.

It is clear that among wages, i.e., the cost of labor, the entrepreneurs' labor income[6] should be included, otherwise the managers'

[5] The wages' quota in the cost of production according to the methods I, II, and III, can also be defined as the wages quota in the "value added by manufacture." The percentages may be calculated from the tables given on the following pages.

[6] That is to say, a compensation for the entrepreneur's labor which might be equalized to the remuneration that he would give to a salaried manager.

salaries would be included in the cost of labor in a limited liability company or a cooperative society and excluded in a firm or a one-man business. The figures obtained from the various kinds of business would then lack uniformity. Moreover, from an economic point of view, the entrepreneurs' labor income also belongs to the cost of labor. There is, however, no unanimity on this point; Dr. J. Marschak does not want to class, in the underlying case, the entrepreneurs' labor income with the other wages (*vide Soziale Praxis*, 4. Jahrgang, Heft 17, page 515). It may, however, be observed that, if the entrepreneurs' labor income is left out, this will make little difference in the undermentioned wages' quota in industrial commodities, as in the majority of cases these commodities are manufactured in establishments that are operated as limited liability companies.

In the following pages the above-mentioned definition of wages will be used. The conception "cost of production" can best be explained by reproducing, in abridged form, the model of the schedules used in the inquiry.

	in guilders
Wages and salaries	.....
Raw materials (the principal materials to be mentioned separately)	.....
Fuel (coal and fuel oils)	.....
Gas, purchased electric energy	.....
Freight	.....
Depreciation	.....
Interest on borrowed and owned capital (in agriculture also rents)	.....
All other charges	.....
Total cost of production	.....

Scope of the inquiry.—The following products and the following industries, collaborating in the production or distribution of these products, have been chosen for the inquiry.

Products	*Industries*
A Workman's tenement	Brick and tile Saw-mills and planing mills Building trade
A book	Paper Printing

	Binding
	Publishing
	Bookselling
Potato-flour	Agriculture (cultivation of potatoes for industry)
	Potato-flour
Pork	Agriculture (pig-breeding and fattening)
	Slaughter-house
	Retail-trade
Vegetables	Horticulture
	Wholesale trade
	Retail trade
Butter	Agriculture (dairy farm)
	Butter-factory
	Retail trade
Sugar	Agriculture (cultivation of beet roots)
	Beet sugar factory
Bread	Agriculture (cultivation of wheat)
	Flour
	Baking
Ready-to-wear woolen suit	Wool-spinning
	Wool-weaving
	Wool-dyeing
	Manufacture of mens' clothing
	Ready-made shops
Cotton shirt	Spinning
	Weaving
	Bleaching
	Dyeing
	Manufacture of men's clothing
	Ready-made shops
Boots and shoes	Tannery
	Manufacture of tanning extracts and solutions
	Boot and shoe factory
	Shoe-shops
Cigars	Wholesale trade in tobacco
	Cigar factory
	Wholesale trade in cigars
	Retail trade
Internal-combustion engines	Blast-furnace
	Foundry
	Internal-combustion engine

Iron Tug-boat	Ship-building
Iron Motor-vessel	Winches Forging Anchors and chains

In all establishments engaged in production as well as in distribution, fuel (including gas and purchased electric energy) is used and freight is paid. For this reason the wages' quota in coal-mining, in gas-works, in electric power stations, and in the transport business, were investigated. The canvass covered the years 1924, 1929, and 1931.

Below, the results of the inquiry will be discussed, separately for the Methods I, II, and III.

Method I. The enterprise wage quota.—As in several cases objections were raised against publication, the composition of the production costs could not be given for all industries included in the investigation. The figures available for publication have been summarized in the following three tables; Table IA comprises agriculture, Table IB the industrial establishments, and Table IC the establishments engaged in distribution.

We should point out the fact that the capital-costs in agriculture (interest and rents) amount to about 30 per cent of the production costs, and, moreover, that the wages' quota increases from 1924 and 1929 to 1931, as a result of the price reduction of the raw materials.

Method II. The commodity wage quota.—We already explained how, by this method, the accumulated wages' quota of the commodities included in the investigation have been ascertained. As far as possible the production costs and the corresponding accumulated wages' quota of the commodity in question have been calculated with relation to the selling value, at the factory as well as at the retailer's shop.

The results of this part of the inquiry have been summarized in Table IIA.

If the investigation could have been all-embracing, the items "Materials b and c" would no more be found in Table IIA. It is clear, however, that an investigation into the wages' quota of all home-made materials is impossible. Nevertheless, it appears from Table IIA that in most cases the Dutch materials, of which the composition of the production costs has not been examined, constitute only a very small part of the total costs of the commodity concerned. The workman's tenement, where the Materials b amount to 14 per cent of the production costs, is an exception; nevertheless it may be assumed that this item contains about 50 per cent of wages.

There is in Table IIA another item of which a Dutch wages' quota must be ascertained, viz., "All other charges." It is a collection of

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TABLE IA.—AGRICULTURE. COMPOSITION OF THE PRODUCTION COSTS IN PERCENTAGES

Type	Agriculture in the clay-ground districts						Agriculture				Dairy farm				Pig-breeding and fattening		Horticulture		
Products cultivated	Wheat			Beetroots			Potatoes for industry				Milk				Hogs		Vegetables		
Year	'23/24	'28/29	'30/31	'23/24	'28/29	'30/31	'23/24	'29/29	'30/31	'31/32	'23/24	'28/29	'30/31	'31/32	'28/29	'30/31	'28/29	'29/30	'30/31
Item																			
Wages	38.0	41.0	40.1	47.7	46.9	48.8	37.3	37.9	39.2	42.1	28.4	28.6	32.1	36.3	9.0	11.5	46.3	45.3	45.8
Materials	5.6	5.3	6.4	15.3	14.1	15.0	25.6[1]	22.0[1]	21.8[1]	19.7[1]	19.5[2]	24.1[2]	19.5[2]	15.1[2]	82.2	78.0	15.7	17.4	16.4
Interest and rents	38.9	35.8	36.6	26.4	27.3	25.0	27.3	29.3	28.5	27.1	33.4	30.5	31.8	31.3	4.8	5.0	23.9	24.5	23.6
Depreciation	4.1	5.1	5.1	2.5	2.9	3.0	3.0	2.8	3.0	3.1	9.6	9.5	9.1	8.9	1.0	1.3	3.8	4.2	4.5
All other charges	13.4	12.8	11.8	8.1	8.8	8.2	6.8	8.0	7.5	8.0	9.1	7.3	7.5	8.4	3.0	4.2	10.3	8.6	9.7
Total	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	10.00	100.00

[1] Artificial manure.
[2] Forage and artificial manure.

TABLE IB.—INDUSTRIES. COMPOSITION OF THE PRODUCTION-COSTS IN PERCENTAGES

Industry	Butter-factory			Potato-flour factory			Flour-mill and bakery (combined)			Beet-sugar factory			Brick-works			Limestone factory[1]			Saw-mill and planing mill		
Products	Butter			Potato-flour			Bread			Sugar			Bricks and tiles						Turned and shaped woodenware		
Year	'24	'29	'31	'24	'29	'31	'24	'29	'31	'24	'29	'31	'24	'29	'31	'24	'29	'31	'24	'29	'31
Item																					
Wages	2.3	2.6	3.8	8.2	6.9	9.7	29.0	29.6	33.1	12.2	13.3	17.5	51.1	57.3	57.7	35.3	40.1	43.9	12.3	12.5	14.9
Materials	88.1	87.95	83.05	69.4	74.9	63.5	55.5	54.4	48.3	60.4	59.4	47.3	5.2	4.0	4.4				75.6	75.1	71.4
Fuel	0.8	0.7	0.95	3.2	1.9	2.8	2.7	2.6	2.7	4.1	4.6	4.3	16.4	11.3	10.6				0.6	0.7	0.6
Freight	3.9	4.4	6.1	5.2	6.0	7.3	1.1	1.3	1.5	6.3	8.0	6.9							1.6	1.8	1.7
Depreciation	1.2	1.2	1.7	5.1	3.1	5.5	2.5	2.7	3.3	8.5	6.7	12.8	6.5	6.5	6.4				1.6	1.8	2.2
Interests	0.9	0.9	1.3	4.7	3.5	5.8	4.2	4.3	4.5	3.9	3.1	5.1	10.3	10.0	9.9				4.2	4.2	4.8
All other charges	2.8	2.25	3.1	4.2	3.7	5.4	5.0	5.1	6.6	4.6	4.9	6.1	10.5	10.9	11.0				4.1	3.9	4.4
Total	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0

[1] Objections have been raised against publication of other figures than the wages' percentage.

TABLE IB.—*Continued*

Industry	Manufacture of window screens, door screens etc.			Building trade[1]	Book-printing			Book-binding			Manufacture of woolen goods[1]			Manufacture of men's clothing[1]			Boot and shoe factory			Cigar factory[1]		
Products	Window and door screens, etc.			Workman's tenement							Textiles			Ready-to-wear suit			Boots and shoes			Cigars		
Year	'24	'29	'31	'29	'24	'29	'31	'24	'29	'31	'24	'29	'31	'24	'29	'31	'24	'29	'31	'24	'29	'31
Item																						
Wages	31.3	28.6	31.8	37.2	66.1	65.5	65.9	54.9	52.5	54.9	25.1	24.6	32.2	28.5	28.0	30.5	27.8	28.8	32.5	45.4	45.8	48.4
Materials	51.0	56.9	53.3		3.5[2]	3.4[2]	2.8[2]	24.0	27.2	23.6							59.9	61.9	55.0			
Fuel	1.0	1.1	1.1		2.4	2.3	2.2	1.8	1.5	1.5							0.9	0.7	0.8			
Freight	1.6	2.0	2.0		0.9	0.8	0.8	0.4	0.7	0.6							1.0	0.95	1.0			
Depreciation	3.8	2.7	3.2		8.8	10.7	11.5	4.5	4.7	5.5							2.0	2.1	2.4			
Interests	4.2	3.2	3.2		11.0	9.8	9.8	6.9	5.9	6.3							2.9	2.6	3.1			
All other charges	7.1	5.5	5.4		7.3	7.5	7.0	7.5	7.5	7.6							4.5	3.95	5.2			
Total	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0

[1] Objections have been raised against publication of other figures than the wages' percentage.

[2] Not including paper and binding.

TABLE IC.—DISTRIBUTION. COMPOSTION OF THE COSTS IN PERCENTAGES

Type	Grocery[1]			Slaughter-house and butcher's shop (combined)	Wholesale dealers' and green-grocers' shops (combined)		Publishing-business			Book-selling			Ready-made shops[2]			Wholesale dealers and cigar stores		
Commodities	Groceries, including butter and sugar			Pork	Vegetables		Books			Books			Ready-to-wear suits			Cigars		
Year	1924	1929	1931	1931	1929	1931	1924	1929	1931	1924	1929	1931	1924	1929	1931	1924	1929	1931
Item																		
Wages	53.5	52.7	55.6	16.2	18.9	18.4	31.4	31.7	32.8	16.0	13.8	15.1	11.6	11.3	11.8	10.6	10.5	10.9
Commodities				70.7	65.9	67.5	53.6[3]	51.4[3]	49.9[3]	71.5	73.1	71.2				76.4	76.9	75.7
Fuel	2.5	2.0	2.2	0.8	0.9	0.8	0.3	0.4	0.3	0.5	0.6	0.7				1.1	0.7	0.8
Freight	7.9	7.9	7.4	1.6	4.1	3.8	0.8	0.8	0.8	1.1	1.0	0.8				0.6	0.6	0.6
Depreciation	5.9	6.9	6.1	2.1	2.1	1.8	0.5	0.5	0.5	2.2	1.6	1.7				1.5	1.2	1.2
Interests	12.9	11.8	10.9	3.0	2.8	2.7	7.1	6.7	7.5	4.1	4.9	5.0				5.2	5.5	6.1
All other charges	17.3	18.7	17.8	5.6	5.3	5.0	6.3	8.5	8.1	4.6	5.0	5.5				4.6	4.6	4.7
Total	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0

[1] Only the cost of distribution.
[2] Materials.
[3] Objections have been raised again. publication of other figures than the wages' percentage.

TABLE IIA.—COMPOSITION OF THE COSTS IN PERCENTAGES. ACCUMULATED WAGES' QUOTA

	Butter						Potato-flour			Bread			Sugar						Pork	Vegetables		A working man's tenement	A book					
	from the selling-value at the																											
	Factory			Shop[1]			Factory			Bakehouse[1]			Factory			Shop[1]			Shop[1]	Shop[1]			Publisher			Shop[1]		
Year	1924	1929	1931	1924	1929	1931	1924	1929	1931	1924	1929	1931	1924	1929	1931	1924	1929	1931	1931	1929	1931	1929	1924	1929	1931	1924	1929	1931
Item																												
Wages and salaries	30.9	31.6	35.6	34.3	34.8	40.2	39.4	40.4	41.4	39.8	40.9	47.3	45.9	47.0	46.2	48.6	50.5	51.2	35.0	53.2	53.4	55.2	60.3	58.5	59.9	59.9	57.4	58.5
Materials																												
a. of foreign origin	15.8	19.7	14.7	14.2	17.5	12.3	17.0	16.0	12.6	32.6	31.7	20.0	10.4	10.5	7.7	8.5	7.5	5.4	33.7	8.7	8.3	20.3	11.5	12.8	10.9	8.5	9.6	7.9
b. origin unknown, but principally of foreign origin	0.5	0.5	0.6	0.4	0.4	0.5	0.9	0.7	0.9	0.4	0.4	0.4	2.2	2.4	2.4	1.9	1.9	1.8	0.2	0.4	0.4	14.0[2]	1.0	1.0	0.8	0.9	0.8	0.7
c. of inland origin										0.9	0.9	0.9											0.6	0.5	0.5	0.4	0.4	0.3
Depreciation	10.0	10.0	9.9	9.6	9.7	9.3	7.9	6.0	8.7	4.4	4.9	5.8	10.7	9.5	15.4	9.8	8.9	12.5	4.8	5.0	5.4	2.1	4.0	4.2	4.7	5.1	4.8	5.1
Interests	31.2	28.7	28.8	29.2	26.8	25 8	23.6	24.8	23.6	13.7	13.0	15.3	20.8	19.8	17.6	19.5	17.7	15.8	13.6	19.4	19.5	3.8	12.3	10.9	11.7	13.1	13.0	13.6
All other charges	11.6	9.5	10.4	12.3	10.8	11.9	11.2	12.1	12.8	8.2	8.2	10.3	10.0	10.8	10.7	11.7	13.5	13.3	12.7	13.3	13.0	4.6	10.3	12.1	11.5	12.1	14 0	13.9
Total	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0

[1] Or in-the-hands-of-the-consumer.

[2] Objections have been raised against publication of other figures than the wages percentage.

[3] Principally home-made materials.

TABLE IIA.—*Continued*

	A woolen[2] ready-to-wear suit						A cotton shirt		Boots and shoes						Cigars[2]						Internal combustion engines			An iron tug-boat			An iron motor-vessel		
	from the selling-value at the																												
	Factory			Shop[1]			Shop[1]		Factory			Shop[1]			Factory			Shop[1]											
Year	1924	1929	1931	1924	1929	1931	1929	1931	1924	1929	1931	1924	1929	1931	1924	1929	1931	1924	1929	1931	1924	1929	1931	1924	1929	1931	1924	1929	1931
Items																													
Wages and salaries	45.2	44.7	51.0	47.4	46.7	52.0	51.6	56.2	33.7	32.5	37.6	37.7	35.8	40.4	48.5	48.7	51.4	48.2	48.5	50.5	49.9	54.0	53.1	42.4	45.2	48.3	45.8	49.5	51.3
Materials																													
a. of foreign origin							17.5	10.4	53.3	55.0	48.1	42.2	44.0	37.6							17.4	12.3	10.0	40.9	39.4	34.6	35.3	33.1	31.5
b. origin unknown, but principally of foreign origin							1.7	1.7													0.5	0.5	0.4	0.2	0.2	0.2	0.4	0.4	0.3
c. of inland origin																													
Depreciation							6.1	7.1	2.8	2.8	3.3	3.6	3.7	4.0							9.3	10.7	10.7	5.3	4.3	4.9	4.4	4.5	4.4
Interests							10.7	11.8	4.5	4.1	4.6	7.8	8.0	8.5							9.4	8.9	11.9	4.9	4.2	5.0	6.4	5.9	6.4
All other charges							12.4	12.8	5.7	5.6	6.4	8.7	8.5	9.5							13.5	13.6	13.9	6.3	6.7	7.0	7.7	6.6	6.1
Total	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0

[1] Or in-the-hands-of-the-consumer.

[2] Objections have been raised against publication of other figures than the wage percentage.

[3] Principally home-made materials.

several smaller items, as cost of repair, advertising costs, travelling expenses, postage, telephone, etc. On the strength of the insight gained through the inquiry by the Central Bureau of Statistics, the Dutch wages' quota in this item may be estimated at 50 per cent on an average.

Thirdly we must take into account the wages' percentage in the item "Depreciation."

When in a certain commodity the accumulated wages' quota forms, e.g., 55 per cent of the production costs, a wages-reduction of 10 per cent would reduce the cost of production by $5\frac{1}{2}$ per cent. But the influence of the wages reduction does not stop here. Nowadays it has gained widespread acceptance that the depreciation of plant and machinery should be calculated from the reproduction-costs of the fixed capital, i.e., the amount for which the capital-instrument (buildings, machinery, etc.), can be replaced. A *general* wages reduction, which will also include the engineering and the building trades, causes a change in the price for building factories and constructing machinery; in other words, the reproduction-cost of the existing capital will be reduced.

As the depreciation of buildings and machinery is calculated on the basis of the reproduction costs, a reduction of wages will reduce the amount that should be reserved for depreciation and will consequently reduce the production costs.

From the investigation it has resulted that the accumulated wages' quotas in the building trade amount to about 65 per cent and in some branches of the engineering trade to about 50 or 60 per cent, so that we can approximately assume that the price of fixed capital (buildings and machinery) contains a wages' quota of about 60 per cent.

Taking account of this fact, we might put 6/10 of the percentage of the production-costs for the wage-factor contained in depreciation.

We should consider, however, that a reduction of costs via the reduced reproduction-costs is, unlike a wages reduction, a secondary effect, which will make itself felt only with a considerable retardation. The entrepreneur is inclined to base his cost accounts on the book-value of the fixed capital as long as possible, and it is only reluctantly that he shifts the basis of his calculations to a lower reproduction value.[7]

[7] The wages reduction also influences the item "Interests" in the production-costs. The fact that the entrepeneur has to pay less wages for the same quantity of products involves a reduction of the floating capital necessary for the payment of wages and this lowers his interest charges. Second, the reduced reproduction-value of the fixed capital means that he has to pay interest on a lower amount of fixed capital, which causes a further reduction of his costs.

The following table gives again the accumulated wages' quota in the finished product, but now including the wages' share in "All other charges," fixed at 50 per cent, and the wages' quota in the item "Depreciation," fixed at 6/10 of the percentage the depreciation forms of the production-costs. The figures relate to the year 1929, as the calculation on Method III has been made for that year only.

TABLE IIB.—ACCUMULATED WAGES' QUOTA

	%		%
Butter	46.0	A ready-to-wear suit (woolen)	54.1
Potato-flour	50.1	A cotton shirt	59.0
Bread	47.9	Boots and shoes	61.4
Sugar	62.6	Cigars	42.3
Pork	44.3	Internal combustion engines	67.2
Vegetables	62.9	An iron tug-boat	51.2
A workman's tenement	65.8	An iron motor-vessel	55.5
A book	67.3		

The difference between Method I and Method II appears clearly from this list; the wages' quota in the butter-factory is only 2.6 per cent, but in butter in-the-hands-of-the-consumer 46 per cent; in the beet sugar factory 13.3 per cent, in sugar in-the-hands-of-the-consumer 62.6 per cent, etc. This shows at the same time how great a difference it makes whether, in a discussion on the consequences of a wages reduction, one method is followed or the other.

Method III. The national wage quota.—The value of the annual production in the Netherlands, in its widest sense, viz., the production and the distribution of all merchandises and services (by private persons and private and public bodies), consists of the following elements:

A. The sum of all wages and salaries.

B. The sum of all *fixed* interests, flowing from capital, invested in the Dutch economic apparatus.

C. The sum of all *variable* interests (rents and dividends), flowing from capital, invested in the Dutch economic apparatus.

D. The sum of all foreign imports (merchandises and services).

On the basis of the statistics of the income tax, the items of the International Balance of Payments, and various kinds of estimates, the items A, B, and C, have been fixed, whereas the item D has been drawn from the external trade statistics.

The result is as follows:

Value of the total annual production in the Netherlands in 1929

	in millions of guilders	in per cent
A. Wages and salaries[8]	4,000	48.8
B. Fixed interests	650	8.0
C. Variable interests	800	9.7
D. Imports	2,750	33.5
	8,200	100.0

It follows from this table that *in 1929 wages and salaries contributed about one-half of the total annual production.*

The large share, ± one third, imports have in the annual production, is very striking. For the U.S.A., e.g., this share amounts to only about 5 per cent.

In the value of the annual production in the Netherlands, no item has been appropriated for depreciation, as it was supposed that the necessary depreciation would be counterbalanced by the improvement and expansion of the fixed capital in the course of the year. Consequently, the wages' percentage of ±50, mentioned above, already comprises the wages that are included in the reproduction-value of the fixed capital. This figure 50 has thus been computed on the same principles as the figures in table IIB.

For the investigation into the costs of the separate commodities (Method II) only those have been chosen for which one or more phases of the process of production take place in the Netherlands. Commodities which pass only through the process of distribution, e.g., imported fruit, imported clothes, were not included in the investigation. But they figure among the value of the total annual production. As in a commodity passing only through the process of distribution the wages' quota will be relatively low, a wages' quota of ±50 in the total annual production will be much lower than the majority of the quota obtained for the separate industries, as will appear from a comparison with Table IIB.

Thus far the Dutch investigation. Moreover, the author tried to collect some data on other countries, which would enable him to compose (or to approximate) for those countries the same calculations as are made for the Netherlands on Method III. For this work special difficulties had to be surmounted, as can be easily understood, and it should be much appreciated if the readers of this paper would kindly draw the attention of the "Central Bureau voor de Statistiek," The Hague, Holland, to corrections in these figures or give information about statistical material which could complete the following data.

[8] Including the entrepreneurs' labor income.

The U.S.A.—The data, can be found in *Recent Economic Changes*, Chapter xii; "The National Income and its Distribution."

Value of the total annual production in the U.S.A., 1925

	in billions of dollars		in per cent
Wages, salaries, pensions, benefits and compensations of employees	46.9		
Entrepreneurs' Labor income	9.5		
Total wages and salaries		56.4	63.7
Rents and royalties	10.6		
Interest (less income from investments abroad)	3.4		
Dividends	4.1		
Corporate surplus (estimation 1924)	2.9		
Entrepreneurial profits	6.9		
		27.9	31.5
Imports (merchandises and services)		4.3	4.8
		88.6	100.0

That the wages' quota in the U.S.A. (past 60 per cent) appears to be so much higher than in the Netherlands (± 50 per cent) can largely be ascribed to the fact that in the former country the imports form such a small part of the value of the annual production.

The United Kingdom.—The figures regarding the national income have been drawn from: Bowley, *The Change in the Distribution of the National Income 1880–1913*, Oxford, 1920.

Value of the annual production in the United Kingdom in 1913

		in millions of £	in per cent
Income from labor		1,355	51.6
Income from property	810		
less: income from investments abroad	200		
		610	23.3
Imports (less re-exports)[9]		659	25.1
		2,624	100.0

The figures relating to the United Kingdom and the Netherlands, especially those regarding the wages' quota, show great conformity, mainly owing to the fact that the figure for the imports is relatively high in both countries.

[9] Société des Nations. *Mémorandum sur le commerce international et sur les balances des payements 1921–1926.*

Germany.—The figures of the national income have been drawn from *Wirtschaft und Statistik* 1932, No. 22.

The value of the annual production in Germany in 1929 and 1930

	1929		1930	
	in millions of R.M.	in per cent	in millions of R.M.	in per cent
Wages and salaries and social charges, not including the entrepreneurs' labor income	45,436	49.8	41,965	51.1
Other income	30,662	33.7	28,200	34.3
Payments of interest abroad (on balance)	800	0.9	825	1.0
Imports (merchandises and services)	14,227	15.6	11,132	13.6
	91,125	100.0	82,122	100.0

Without the entrepreneurs' labor income the wages' quota amounts to about 50 per cent. The question arises whether it is possible to approximate the entrepreneurs' labor income. Among the "other income" ad R.M. 30,662 millions in 1929 there are two items in which the entrepreneurs' labor income is comprised: viz., the income derived from "Land-und Forstwirtschaft"[10] ad R.M. 5,774 millions and the income from "Handel und Gewerbe"[10] ad R.M. 11,768 millions. In *Recent Economic Changes* it is estimated that the entrepreneurs' labor income amounts to 58 per cent of the total entrepreneurs' income; according to Professor Bowley's calculations it would amount to over 60 per cent.

Taking, with regard to Germany, two-thirds ($66\frac{2}{3}$ per cent) as entrepreneurs' labor income, so as to estimate the wages' quota not too low, then the amount of the wages and salaries will be increased (by two-thirds of R.M. 5,774 millions+R.M. 11,768 millions) i.e., by R.M. 11,695 millions and will reach the total amount of R.M. 57,131 millions on a total annual production of R.M. 91,125 millions, that is, a wages' quota of 62.7 per cent.

Though only rough estimates have been made, one can nevertheless conclude from this figure that Professor Zahn's and Adolf Weber's supposition that wages would constitute 80 or 90 per cent of the value of products cannot be maintained.

The data collected on Method III can moreover give an insight into another problem, viz., the proportion of the national income that is attributable to capital and labor. With the aid of the above-men-

[10] The income of the independent entrepreneurs.

tioned figures the following table can be drawn up for the three countries.

	United Kingdom 1913		*United States 1925*		*The Netherlands 1929*	
	in millions of £	in per cent	in billions of $	in per cent	in millions of *f*	in per cent
Income from labor	1,355	62.6	56.9	67.1	4,000	68.4
Income from capital	810	37.4	27.9	32.9	1,850[11]	31.6
	2,165	100.0	84.8	100.0	5,850	100.0

The share derived from property is greatest for England; as to the U.S.A. and the Netherlands, here the proportions are practically the same.

The Hague, Holland

[11] From investments abroad *f*400 millions.

PRICE ANALYSIS

SELECTED REFERENCES ON THE THEORETICAL ASPECTS OF SUPPLY AND DEMAND CURVES AND RELATED SUBJECTS

Compiled by LOUISE O. BERCAW
under the Direction of MARY G. LACY, Librarian
Bureau of Agricultural Economics

THIS list is a revision and enlargement of the first section of a bibliography on *Price Analysis; Selected References on Supply and Demand Curves and Related Subjects* issued by the Bureau of Agricultural Economics of the United States Department of Agriculture in September, 1933, as *Agricultural Economics Bibliography No. 48*. The aim has been to include only references to material published since 1927. However, a few publications issued prior to that date have been included. The original bibliography and also two earlier bibliographies, *Factors Affecting Prices* (issued in March, 1926, as *Agricultural Economics Bibliography No. 14*) and *Farmers' Response to Price*, compiled by O. V. Wells, issued in April, 1933, may be obtained from the Bureau of Agricultural Economics.

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE. Reports of the fourth Cleveland meeting . . . *Science* **73** (1884): 137–168. Feb. 6, 1931.—In the report of Section A (Mathematics) and related organizations, p. 150, there is a short report of a paper by C. F. Roos as follows: "C. F. Roos, of Cornell University, gave an ordered account of the more recent mathematical formulations expressing the interrelations of demand, cost of production, and profit, he showed by concrete examples what may be accomplished in fitting mathematical functions to actual commodity curves over a number of years."

[AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE.] The Syracuse meeting of the American association for the advancement of science and associated societies; edited by Dr. Charles F. Roos. *Science* **76** (1960): 63–75. July 22, 1932.—The following is quoted from the report of Section K, pp. 70–71: "Mr. R. H. Whitman, of the University of Chicago, opened the Tuesday morning session, devoted to a symposium on 'Demand,' with an account of his work on the fitting of demand functions of more general types than those used by H. L. Moore to price data for iron and steel. Terms involving the rate of change of price and the time-integral of price were included in various combinations. High multiple correlations were obtained for each of the periods 1902–1915, 1916–1920, and 1921–1930. Dr. C. F. Roos showed how the effect of past prices and the memory of past purchases lead to a demand function containing an integral either of past prices or past purchases, the two forms being equivalent on account of the invertibility of the linear integral equation. He declared that in the newer view of demand the concept of utility will be demoted from a major to a very minor place. . . . The Thursday afternoon symposium on agricultural supply and demand functions was opened by Professor F. A. Pearson, of Cornell University, who gave the results of a study of the relations between the supplies of corn, potatoes, wheat, and beef, in various localities of the United States and prices (in some cases retail and in others wholesale prices) in New York City. In the next paper Dr. Mordecai Ezekiel [*q.v.*] examined various functional setups for the demand curves of two competing products. In particu-

lar, he discussed the cases where the price of each product is expressed as an additive function of the supply of both, and where the price of each is expressed in terms of its own supply and the price of the other. In the last paper of this symposium, Dr. Louis H. Bean [*q.v.*] of the Department of Agriculture, discussed the derivation of instantaneous demand-supply curves, characteristics of shifts in demand curves, and a type of demand-supply curve which represents the response of producers to price in future planning."

AMOROSO, LUIGI. La curva statica di offerta. *Giornale degli Economisti e Rivista di Statistica* **45** (1): 1–26. Jan. 1930.—The static curve of supply. Abstracted in *Social Science Abstracts*, v. **3**, Nov. 1931, item 9182.

AMOROSO, LUIGI. Ecuaciones diferenciales de la dinâmica econômica. *Revista Nacional de Economia* **28** (84): 241–253. Mar.–Apr. 1929.

AMOROSO, LUIGI. Equazione differenziale della domanda e teoria matematica delle crisi economiche. *Giornale degli Economisti e Rivista di Statistica* **45** (i.e. 46) (1): 39–40. Jan. 1931.—"The criticism of the author's theory, in the definition of the curves of demand and supply made by C. F. Roos (*J. Pol. Econ.* Oct. 1930, 502) is based on an error of interpretation. The theory advanced by Roos on the cyclical movements of quantities and prices as integrals of a differential equation is less simple than that advanced by the author more than a year earlier."—*Social Sci. Abs.*, v. **4**, June 1932, item 9537.

AMOROSO, LUIGI. Le equazioni differenziali della dinamica economica. *Giornale degli Economisti e Rivista di Statistica* **44** (2): 68–79. Feb. 1929.—"In this article the author discusses the contributions of the American economists, G. C. Evans, C. F. Roos, H. L. Moore, and I. Fisher, to the renaissance of mathematical economics. Their theories are evaluated and criticized. Special attention is given to the problem of elasticity of demand and to the work of Fisher for the mathematical prevision of price oscillation."—*Social Sci. Abs.* v. **1**, Dec. 1929, item 10125.

AMOROSO, LUIGI. Intorno all determinazione empirica delle leggi della domanda e dell'offerta. *Giornale degli Economisti e Rivista di Statistica* **45** (11): 941–944. Nov. 1930.—"It is necessary to proceed simultaneously from the empirical data of prices and consumption to the curves of demand and supply for a theoretical solution of the problem. Interpolating the data of the historical curve by means of differential equations it is possible to reach, on the basis of certain hypotheses, the two families of demand and supply curves."—*Social Sci. Abs.* v. **4**, Apr. 1932, item 5576.

BAILEY, A. L. The analysis of covariance. *Jour. Amer. Statis. Assoc.* **26** (n.s. 176): 424–435. Dec. 1931.—"Four examples have been chosen to show some of the practical applications of the analysis of covariance as well as to illustrate the wide variety of information which can be obtained in particular cases. These examples all present imaginary observations but are parallel to actual problems to which the same methods have been applied advantageously. They are taken from the fields of biometry, agronomy, market analysis, and economic supply and demand. In the last of these problems all of the original data and calculations are given in detail. It also illustrates how closer estimates of the correlation and regressions can be obtained by second or third approximations in those cases where the variation between groups is continuous."

BEAN, L. H. Applications of a simplified method of graphic curvilinear correlation. A preliminary report. 2 pts., mimeogr. Washington, D.C., U.S. Department of Agriculture, Bureau of Agricultural Economics, Division of Statistical and Historical Research, 1929.—The following is quoted from p. 1: "The purpose of this paper is to present this simple approach to multiple curvilinear correlation. The method employed will be demonstrated with six examples, or cases, of actual problems chosen from different fields of economic relationships, in the hope that this 'case' method of presentation will not only make clear the simple steps involved, but will also suggest their application to similar problems likely to be encountered by the analyst of variations in economic data. It will be demonstrated by means of a generalized problem, but in this final illustration also we shall refrain from generalization.

Technical language will be used as little as possible, but the reader will need to study closely the graphic presentations, for the method is essentially one of graphic correlation. In each example the data used and the steps in the analysis will be so indicated that the reader may properly appraise the reasonableness of the approach and the reliability of the results. The assumptions and logic involved in each of the six special cases will also be indicated but this only briefly." Case I, pp. 2–4: Relation of (1) production of early potatoes and (2) the price of old potatoes to (3) the price received by producers of early potatoes. Case II, pp. 4–5: Relation of cotton prices and business conditions to the domestic mill consumption of cotton. Case III, pp. 5–7: Cotton consumption (continued). Case IV, pp. 7–8: Effect of supply and other factors on the yearly average of farm price of apples. Case V, pp. 8–9: Effect of supply and other factors on the New York price of California oranges. Case VI, pp. 9–10: Effect of price on acreage of cotton harvested in the United States. Part II is entitled "The Method Applied to Changes in Acreages, Yields, and Livestock Numbers." Three examples, or cases, are given. Case VIII, pp. 1–2: The relation of price to changes in the United States acreage of cabbage. Case IX, pp. 3–4: The relation of price to changes in the number of hogs on farms in the United States on January 1.

BEAN, L. H. Characteristics of agricultural supply and demand curves. 6 pp., 11 pp. of charts, mimeogr. [Washington, D.C.] United States Department of Agriculture, Bureau of Agricultural Economics, Division of Statistical and Historical Research. [1933].—This is a summary of an address before Section K of the American Association for the Advancement of Science, in a joint program with the Econometric Society, Syracuse, N.Y., June 22, 1932. The following is quoted from the writer's introductory paragraphs: "This paper attempts to bring together a number of studies, made by the writer during the past three or four years, which have been of practical use in anticipating changes in prices, marketings, and subsequent farm production. They are an outgrowth of the price analysis work sponsored by the Bureau of Agricultural Economics in its attempts to supply farmers with outlook information relative to prospective changes in supply and demand conditions. The topics covered . . . are (1) a simple illustration of an approach to the instantaneous supply and demand curves for commodities of fixed annual supply. The supply curves dealt with . . . are the current-supply curves representing supplies offered from a total fixed available quantity on hand, (2) relations between producer, dealer, and total demand curves and two views of price determination, (3) shifts in demand curves, that involve changes in slope (of elasticity) and changes in level due to the value of money, population growth, etc., (4) comparison of agricultural with non-agricultural demand curves, and (5) subsequent-supply curves, which show the effect of current supplies of the following season or seasons. The method of analysis used . . . is the simplified graphic approach to multiple curvilinear correlation described by the writer in the *Journal of the American Statistical Association* in December, 1929, and December, 1930." Commodities for which curves are given are potatoes, pig iron and steel, meat animals, and hogs.

BEAN, L. H. The farmers' response to price. *Jour. Farm Econ.* **11** (3): 368–385. July, 1929. Issued also as a reprint.—"Summarizes the results obtained from statistical analyses of the relations between prices received by producers and subsequent changes in the acreages of potatoes in New York, Michigan, Idaho, and the United States, of flax in North Dakota, and of sweet potatoes, cabbage, strawberries, watermelons, rye, and cotton, and in the number of hogs in the United States."—Wells, O. V. Farmers' response to price, item 1.

BEAN, L. H. Measuring the effect of supplies on prices of farm products. *Jour. Farm. Econ.* **15** (2): 349–374. Apr. 1933.—Discussion by H. A. Wallace, pp. 374–377.

BEAN, L. H. A simplified method of graphic curvilinear correlation. *Jour. Amer. Statis. Assoc.* **24** (n.s. 168): 386–397. Dec. 1929.—In chart IV, p. 395 "is an analysis of the price of early potatoes received by growers, showing the effect of production of early potatoes and of the competitive effect of the price of old potatoes for the years 1921–1928 inclusive." In chart V, p. 396 "is an analysis of the effect of price and business activity on the domestic mill consumption of cotton for the years 1919 to 1928, inclusive." A further application of this method by Mr. Bean is given in Application of a Simplified Method of Correla-

tion to Problems in Acreage and Yield Variations in *Jour. Amer. Statis. Assoc.* **25** (n.s. 172): 428–439. Dec. 1930.

BEAN, L. H., and THORNE, G. B. The use of "trends in residuals" in constructing demand curves. *Jour. Amer. Statis. Assoc.* **27** (177): 61–67. Mar. 1932.

BLACK, J. D. Agricultural reform in the United States. 511 pp. New York, London, McGraw-Hill Book Company, Inc., 1929.—The effect of surpluses on prices and incomes, ch. IV, pp. 85–128. Subtopics are: Weather surpluses, overplanting surpluses, relation of supplies to prices, effect of surpluses and deficits on gross values of crops, effect of surpluses on net value of crops, effect of prices on subsequent production, surpluses of livestock and livestock products. Season surpluses and gains from holding during the year, ch. VI, pp. 152–176.

BLACK, J. D. Elasticity of supply of farm products. *Jour. Farm Econ.* **6** (2): 145–155. Apr. 1924.

BORDIN, ARRIGO. La legge della domanda dal punto di vista della statica e della dinamica. *Giornale degli Economisti e Rivista di Statistica* **45** (5): 421–471. May 1930.—The law of demand from a static and dynamic standpoint. Abstracted in *Social Science Abstracts*, v. **3**, Dec. 1931, item 18856.

BROWN, E. H. P. Demand functions and utility functions: a critical examination of their meaning. ECONOMETRICA **2** (1): 51–58. Jan. 1934. Based upon a paper read before the meeting of the Econometric Society, Paris, October, 1932. —"The statistical derivation of demand functions has now become a familiar part of econometric procedure: passing beyond the rough estimates and unfilled projects with which it began, it has been implemented in a large number of statistical studies. In most of this work the functions have been defined in terms of real price; but in the studies of Professors Fisher and Frisch certain methods have also been devised which proceed directly to 'the measurement of marginal utility.' The critical discussion which this work has occasioned has been concerned in part to ask how far the methods used can yield results of the kind sought, and in part to ask what kind of result it is that should be sought. It is with the second question that this paper is concerned. The paper begins by giving reasons for believing that the measurement of marginal utility must not be conceived as a measurement of satisfactions; it then considers the formulation of an index of disposition which will embody the essential principle of the marginal utility function without being exposed to the same philosophical objections; and finally it examines the properties of the real price function in comparison with those of the proposed index."

BURGESS, R. W. A statistical approach to mathematical formulation of demand-supply-price relationship. *Ann. Math. Statis.* **3** (1): 10–19. Feb. 1932.—"A generalized mathematical procedure is desirable, involving cost, demand, and supply functions, to analyze more complex situations than are amenable to present methods. The non-statistical mathematical economist, before formulating equations with undetermined constants (the determination of which is usually at least as difficult as the original problem), needs to give more attention to the assumptions implicit in the form of equations used. Scientific procedure in supply-demand-price analysis cannot neglect the statistical phases of the problem. The development of actuarial science illustrates broadly the necessary steps in scientific progress; (1) creation of a serviceable mechanism of measurement, (2) careful making and recording of numerous data, (3) derivation of empirical laws from these data, and finally, (4) discovery of fundamental principles."—*Social Sci. Abs.*, v. **4**, Oct. 1932, item 16585.

BYE, R. T. Composite demand and joint supply in relation to public utility. *Quart. Jour. Econ.* **44** (1): 40–62. Nov. 1929.

BYE, R. T. The inductive testing of an economic deduction: a method illustrated by analyses of marginal producers and representative firms, by Alfred Marshall, Horace Secrist, and Kemper Simpson. Analysis No. 44 in *Methods in Social Science*, edited by Stuart A. Rice. Chicago, University of Chicago Press, 1931, pp. 614–634.—"A description of the attempt of Secrist and Simpson

to obtain a statistical verification of the theoretical cost curve of Marshall."—Wells, O. V. Farmers' response to price, item 5.

CABIATI, ATTILIO. Sull' influenza della elasticità della domanda nella produttività crescente. *Giornale degli Economisti e Rivista di Statistica* **45** (4): 335–337. Apr. 1930.—"The elasticity of demand often acts as a limit to the possibility of expansion of an enterprise. Therefore, even if the enterprise is in the phase of increasing productivity, it is often not possible to realize the whole course of its expansion because its maximum advantage up to the point of monopoly will be in proportion as it is able to realize internal economies."—*Social Sci. Abs.*, v. **3**, 1931, item 9187.

CASSELS, J. M. A critical consideration of Professor Pigou's method for deriving demand curves. *Econ. Jour.* **43** (172): 575–586. Dec. 1933.—A brief reply by A. C. Pigou is made on pp. 586–587.

CASSELS, J. M. The nature of statistical supply curves. Address delivered before the American Farm Economics Association, December 1932. *Jour. Farm. Econ.* **15** (2): 378–387. Apr. 1933.—"An analysis of the statistical approach to the price-supply problem. It is pointed out that the slope of the supply curve depends on the time allowed for adjustments to take place and that the contraction response is not an exact reversal of the expansion response."—Wells, O. V. Farmers' reponse to price, item 6.

CLOWER, F. W. Note on the supply curve for capital. *Amer. Econ. Rev.* **18** (2): 272–274. June, 1928.

COPELAND, M. A. Economic theory and the natural science point of view. *Amer. Econ. Rev.* **21** (1): 67–79. 1931.—"This paper, presented at the forty-third annual meeting of the American Economic Association, Cleveland, Ohio, December 30, 1930, considers how standards of social policy, the marginal utility theory and the ultimate determinants of price, the law of supply and demand, the productivity theory, Say's law that real supply is real demand, and theory of taxation, might be treated from a natural science point of view."—*Expt. Sta. Rec.* **65**: 479. 1931.

DOMINEDÒ, VALENTINO. Ancora in tema di curve di domanda. *Giornale degli Economisti e Rivista di Statistica* **49** (2): 115–116. Feb. 1934.

DOMINEDÒ, VALENTINO. Considerazioni intorno alla teoria della domanda. *Giornale degli Economisti e Rivista di Statistica* **48** (1, 11): 30–48, 765–807. Jan. 1933, Nov. 1933.—This paper presenting some considerations on the theory of demand is in two parts. Part I is Rilievi critici sulla determinazione del prezzo [critical remarks on the determination of price] and Part II is Le principali premesse e caratteristiche delle curve statiche [the principal premise and characteristics of the statistical curve].

DOUGLAS, P. H. Elasticity of supply as a determinant of distribution. In *Economic Essays Contributed in Honor of John Bates Clark*, ed. by Jacob H. Hollander, pp. 70–118. New York, The Macmillan Co., 1927.—Titles of the ten parts of this paper are: The positive contributions of the marginal theory of distribution; The place of supply curves of the factors in a complete theory of distribution; The conscious or unconscious use of supply schedules in economic theory; Various types of supply curves and the meaning of elasticity of supply; Elasticities of supply in relation to increases in the effectiveness of industry; Elasticity of supply in relation to decreases in the net effectiveness of industry; Elasticities of supply in relation to changes in bargaining power; The influence of the relative proportion of the total product received by the factors; Other factors; Some next steps in research.

ECONOMETRIC SOCIETY. The meeting of the Econometric Society in Chicago, June, 1933. ECONOMETRICA **1** (4): 431–444. Oct. 1933.—Pages 431–433 of this report, by Alfred Cowles, 3rd, are devoted to a summary of a joint session of the Econometric Society and Section K of the American Association for the Advancement of Science. The general subject was Elasticities of Demand and Supply. Papers read were Elasticity of Supply for Agricultural Products by Sir Daniel Hall; Elasticity of Supply for Agricultural Products, with Special

Application to Corn and Potatoes by Harry Pelle Hartkemeier [published as *Missouri University Studies*, v. 7, no. 4, October, 1932]; Elasticity of Demand for Producers' Goods as Illustrated by the Demand for Iron and Steel, by Roswell Whitman; and The Elasticity of Demand for Cotton, by L. H. Bean. A report of this session is also given in *Science* n.s. **78** (2013): 80–81. July 28, 1933.

ECONOMETRIC SOCIETY. The meeting of the Econometric society in Cincinnati, Ohio, December 1932. ECONOMETRICA **1** (2): 209–217. Apr. 1933.—The report of this meeting is by Alfred Cowles, 3rd. A summary and comments by M. D. Anderson, H. T. Davis, and W. F. Ferger, on Henry Schultz' paper Interrelations of Demand [*q.v.*] pp. 210–212. Extracts from Victor S. von Szeliski's paper, The Statistical Analysis of Stock Prices, pp. 214–215.

ECONOMETRIC SOCIETY. The meetings of the Econometric Society in Philadelphia and Boston, December 1933. ECONOMETRICA **2** (2): 204–220. Apr. 1934.—This report is by Alfred Cowles, 3rd. Among the papers for which short summaries are given are the following: Monopolistic Competition and the Productivity Theory of Distribution, by E. H. Chamberlain; Imperfect Competition—A Problem in General Equilibrium, by William Jaffé; Imperfect Competition: Some Aspects of Cost and Demand, by Theodore O. Yntema; Studies in Demand During Periods of Rapid Economic Changes, by E. J. Working [*q.v.*; no summary given]; Times Series and the Derivation of Demand and Supply Curves (A Study of Coffee and Tea, 1850–1930), by Elizabeth W. Gilboy.

ECONOMETRIC SOCIETY. [Reports of meetings, 1931 and 1932] ECONOMETRICA **1** (1): 73–86, 87–90, 91–93, 94–104. Jan. 1933.—25a. La réunion de la Société d'Économétrie, Lausanne, septembre, 1931, by Hans Staehle, pp. 73–86. On the morning of September 23 M. Marschak gave a paper on "La Détermination statistique des courbes d'offre et de demande, exposé général." A short report of the discussion of this paper is given on pp. 80–81. On p. 81 mention is made of papers by M. Fanno on "Les correlations des prix et les courbes statistiques de demande et d'offre" [*q.v.*] and by R. Roy on "La demande dans ses rapports avec la distribution des revenus." [*q.v.*] 25b. The meeting of the Econometric Society in Washington, D.C., December 1931, by Irving Fisher, pp. 87–90. Short accounts of the papers read at the meetings are given on pp. 88–90. Among the papers reported are the following: The Relation between Mathematical Economics and Statistical Price Analysis, by Mordecai Ezekiel; Some Relations between Commodity Prices and Business Activity by Holbrook Working; and A Comparison of the Elasticities of Demand for Selected Commodities, Obtained by Different Methods, by Henry Schultz [*q.v.*]. 25c. The meeting of the Econometric Society in New Orleans, Louisiana, January 1932, by Irving Fisher, pp. 91–93. The following is quoted from the short report of the meeting on p. 92: "... The following paper was that of Professor Hotelling on 'Edgeworth's Taxation Paradox and the Nature of Demand Functions,' which discussed certain surprising results of two interrelated demands when one of them was subjected to a tax. The discussion of this paper was led by Dr. Mayer and Professor Rietz ... At noon, a luncheon meeting was held ... The following papers from Europeans were read by title: ... 2. 'Pitfalls in the Statistical Construction of Demand and Supply Curves,' by Ragnar Frisch [*q.v.*]. 3. 'Cio' Che e' Chiaro e Cio' Che e' Oscuro nelle Fluttuazioni dei Prezzi,' ('Clear and Dark Aspects in Price Fluctuations'), by Luigi Amoroso (Published in Vol. 4 of *Atti dell' Istituto Nazionale delle Assicurazioni*). 4. 'The Demand for Rival (or Substitute) Commodities,' by Umberto Ricci [*q.v.*]. 5. 'Kostenanalyse als Grundlage einer statistischen Ermittlung von Nachfragekurven,' by Erich Schneider (published in *Archiv für Sozialwissenschaft*) ..." 25d. The meeting of the Econometric Society, in Syracuse, New York, June 1932, by Joseph Mayer, pp. 94–104. A Demand and Supply symposium was held on the morning of the first day of the meeting. A short abstract of the first paper, The Statistical Law of Demand for Producers' Goods as Illustrated by the Demand for Iron and Steel, by Roswell H. Whitman, together with a report of the discussion on this paper by Harry S. Kantor, Ragnar Frisch, Harold Hotelling, and L. H. Bean, is given on pp. 94–96. A short abstract of C. F. Roos' paper, The Effect of Hysteresis and Speculation on Demand and Supply and a report of discussion on this paper by Joseph Mayer, Mordecai Ezekiel, Harold Hotelling, H. T. Davis, and Ragnar Frisch, are given on pp. 96–97. A short report of L. L. Thurstone's paper, An

Experimental Study of Indifference Curves, together with discussion by Dr. Ezekiel and Dr. Mayer, is given on pp. 97–98. Short accounts are given on pp. 102–103 of three papers presented at the Wednesday afternoon session, i.e.: Supply-Price Relationships, by F. A. Pearson and G. F. Warren; Analyzing the Prices of Competing Products, by Mordecai Ezekiel; and Characteristics of Agricultural Supply and Demand Functions, by L. H. Bean [*q.v.*].

ENGLÄNDER, OSKAR. Elastizität der nachfrage. *Schmöllers Jahrbuch für Gesetzgebung, Verwaltung und Volkswirtschaft im Deutschen Reiche* **53** (3): 1–28. June 1929.—"A discussion on measuring elasticity of demand under conditions of changing quantity on changing price . . . " *Social Sci. Abs.*, v. **2**, Jan. 1930, item 473.

EZEKIEL, MORDECAI. Preisvoraussage bei landwirtschaftlichen erzeugnissen. *Frankfurter Gesellschaft für Konjunkturforschung. Veröffentlichungen* . . . hft. 9, 32 pp. Bonn, K. Schroeder, 1930.—"For a commodity with a definite sales period within which the supply is fixed, the price for that period may be estimated as soon as the supply is known, from the observed relation of supply to price in earlier periods. Annual crops such as cotton fulfill this condition. The change in the value of money and trends in demand may need to be included in the analysis. Once the forecasted average price for the season has been determined, the seasonal price movement may be forecasted on the basis of average seasonal movements in previous years of similar characteristics. Thus the usual seasonal movement of potato prices differ in years of small, medium, and large crops. The probable influence of price upon consumption and upon carry-over may similarly be determined by comparing previous changes in these factors with changes in prices for the same periods. The changes in quantities consumed and in quantities added to or subtracted from carry-over may be added together to give a total demand curve, to check against the curve determined directly from changes in supply and price. The influence of price upon consumption may likewise be measured by comparing the changes in acreage planted with changes in price received one and sometimes two or more seasons earlier. More elaborate and complicated methods of determining the relations may be used, including multiple correlation and logarithmic transformations. Other factors may need to be considered. Prices vary in time, in space (geographically), and in different stages of the marketing process. Each type of variation may be subjected to statistical and economic study."—*Social Sci. Abs.*, v. **4**, Oct. 1932, item 17638.

EZEKIEL, MORDECAI. Some considerations on the analysis of the prices of competing or substitute commodities. ECONOMETRICA **1** (2): 172–180. Apr. 1933.—Paper read before a joint meeting of the Econometric Society and Section K of the American Association for the Advancement of Science, Syracuse, New York, June, 1932.

EZEKIEL, MORDECAI. Statistical analyses and the 'laws of price.' *Quart. Jour. Econ.* **42** (2): 199–227. Feb. 1928. Also issued as a reprint.

EZEKIEL, MORDECAI. A statistical examination of the problem of handling annual surpluses of non-perishable farm products. *Jour. Farm. Econ.* **11** (2): 193–226. Apr. 1929. Also issued in reprint form. Also issued in mimeographed form by the U. S. Bureau of Agricultural Economics.—"A discussion of ways of dealing with farm surpluses may consider either the institutional means which might be established for disposing of surpluses, or the possibilities of increasing returns to farmers by disposing of the surpluses in various ways. This paper will be confined to the second problem. The possible effects on returns to farmers of such operations as storing from year to year and export dumping will be considered in the light of what is known about price behavior from statistical studies of prices."—p. 193. Unpublished price-analysis studies of the following were used in the preparation of this paper: E. M. Daggit, studies of factors affecting wheat prices; L. H. Bean, a study of the probable effects of export dumping of wheat; Owen L. Dawson, studies of the export demand for various pork products. The computations on cotton are based on studies by the author and by L. H. Bean.

FANNO, MARCO. Correlazioni tra prezzi e curve statistische di domanda e offerta. *Rivista Italiana de Statistica, Economia e Finanza* **4** (2): 223–237. June 1932.—Presented at the September 1931 meeting of the Econometric Society.

Also published in ECONOMETRICA **1** (2): 162–171. Apr. 1933 with title: Interrelations des Prix et Courbes Statistiques de Demande et d'Offre.

FANNO, MARCO. Die elastizität der nachfrage nach ersatzgütern. *Zeitschrift für Nationalökonomie* **1** (1): 51–74. May 1929.—"The author remarks that, while Alfred Marshall has developed the notion of elasticity of demand and given its mathematical expression, he did not apply it to the particular case of the demand for substitution goods. The author endeavors to complete the theory at this point. He examines the principles for the use of substitution goods and studies which influence its elasticity and finally develops the Marshallian formula adapted to the particular case under consideration."—*Social Sci. Abs.*, v. **2**, Nov. 1930, item 14449.

FERGER, W. F. Notes on Pigou's method of deriving demand curves. *Econ. Jour.* **42** (165): 17–26. Mar. 1932.—Page 26 contains also a brief reply by Mr. Pigou. Abstracted in *Social Science Abstracts*, v. **4**, Sept. 1932, item 15826.

FERGER, W. F. The static and dynamic in statistical demand curves. *Quart. Jour. Econ.* **47** (1): 36–62. Nov. 1932.

FISHER, IRVING. A statistical method for measuring "marginal utility" and testing the justice of a progressive income tax. In *Economic Essays Contributed in Honor of John Bates Clark*, ed. by Jacob H. Hollander, pp. 157–193. New York, The Macmillan Co., 1927.—"The method consists, in a word, of so utilizing data of family budgets and prices as to compare the wants of two typical families of different incomes, in the same community, by using as a yardstick or criterion a third typical family having identical tastes, but differing in the amount of income, and living under a different scale of prices for foods, rents, clothing, and other items of consumption."—p. 159.

FOÀ, BRUNO. In tema di curve di domanda. *Giornale degli Economisti e Rivista di Statistica* **48** (8): 571–579. Aug. 1933.

FRISCH, RAGNAR. New methods of measuring marginal utility. Tübingen, J. C. B. Mohr, 1932. (Beiträge zur ökonomischen theorie, 3).—Reviewed by Henry Schultz in Jour. Polit. Econ. **41** (1): 95–116. Feb. 1933 in an article entitled Frisch on the Measurement of Utility. Also reviewed by A. L. Bowley in *Econ. Jour.* **42** (166): 252–256. June 1932.

FRISCH, RAGNAR. Pitfalls in the statistical construction of demand and supply curves. *Frankfurter Gesellschaft für Konjunkturforschung. Veröffentlichungen*, n.f., hft. 5, 39 pp. Leipzig, H. Buske, 1933.—Text in English. A discussion of the pitfalls connected with "the variability type of the data" used in constructing supply and demand curves. Reviewed by Umberto Ricci in *Weltwirtschaftliches Archiv* **39** (2): 104–105. Mar. 1934.

FUBINI, RENZO. Sull'influenza dell'imposta sulla domanda e sull'offerta. *Giornale degli Economisti e Rivista di Statistica* **44** (1): 12–22. Jan. 1929. The influence of taxation on supply and demand.

GILBOY, E. W. Demand curves by personal estimate. *Quart. Jour. Econ.* **46** (2): 376–384. Feb. 1932.

GILBOY, E. W. Demand curves in theory and practice. *Quart. Jour. Econ.* **44** (4): 601–620. Aug. 1930.—Abstracted in *Social Science Abstracts* v. **3**, Jan. 1931, item 625.

GILBOY, E. W. The Leontief and Schultz methods of deriving "demand" curves. *Quart. Jour. Econ.* **45** (2): 218–261. Feb. 1931.—"Starting with a brief description of the technique of the methods developed by Schultz and Leontief for deriving statistical demand and supply curves the author proceeds to an examination of the assumptions and logic upon which their conclusions rest and then applies their methods to the derivation of demand curves, and coefficients of elasticity from price quantity data for sugar, copper, and coffee, for several price and quantity import and export index numbers for England. Although both Schultz and Leontief derive average demand and supply curves purporting to measure the average elasticity over a given interval of time and the shiftings of

instantaneous schedules for given years within the period from the average schedules for the entire period, the author questions their validity. The Schultz curve she holds is a demand curve only in case supply can be disregarded as an independent factor, or supply alone moves, and the Leontief curve does not measure demand unless his hypotheses are found to be true to fact for the commodity in question. In a parallel comparison of curves and coefficients of elasticity she demonstrates that the results obtained from the two methods are distinctly different and concludes that neither yields the static demand curves discussed by Marshall."—*Social Sci. Abs.*, v. 3, July 1931, item 10863.

GRAAFF, A. de Grundsätzliches zur messbarkeit der nachfrage-elastizität. *Weltwirtschaftliches Archiv* 39 (1): 94–150. Jan. 1934.

GREEN, R. M. An application of "movement theory" in the behavior of prices to corn prices. *Jour. Farm. Econ.* 14 (2): 358–361. Apr. 1932.

GUPTA, AMIYA KUMAR DAS. A note on the elasticities of demand and supply. *Indian Jour. Econ.* 13 (4): 676–684. Apr. 1933.

HARROD, R. F. Notes on supply. *Econ. Jour.* 40 (158): 232–241. June 1930. —"Propaedeutic to the intricate questions connected with external and internal economics which have recently received important treatment by Pigou (*Econ. J.*, June 1928) and by Shove (*Econ. J.*, Mar. 1930), Harrod attempts to demonstrate mathematically the relationship between the supply schedule and increasing and decreasing costs. He does this (1) with reference to a short period, (2) with reference to a long period. No definite conclusions are drawn with regard to the short period but analysis of the effects of varying changes in demand at a given point in time is made. With reference to a long period, the whole problem is rendered complex due to the introduction of dynamic factors and particularly the introduction of the equilibrating of the factors of production with changes in demand. In general, he concludes that (1) an increase in demand would reduce the short-period prime cost of production and consequently the supply schedule of the industry; (2) an increase in demand would also reduce the average cost of production if all kinds of overhead charges are included; (3) notwithstanding this an increase in demand would raise the price of the product, and consequently the industry is subject to the 'law of increasing costs' as commonly understood. 'Confusion has arisen through failure to observe that these propositions are mutually consistent.' "—*Social Sci. Abs.*, v. 3, Jan. 1931, item 626.

HAY, W. W. Study of the nature of demand would obviate many of the mistakes of management. *Annalist* 37 (957): 931–932. May 22, 1931.—"Inability to forecast demand . . . remains one of the main difficulties in the way of early adjustment to orderly, balanced production. There are no easy yardsticks by which to measure demand in the sense that we can measure supply, production, shipments, stocks, etc. Nevertheless, a correct adjustment presumes some degree of balance between capacity to produce (supply) and the market (demand). It is therefore necessary to consider the influence of relative market saturation on demand." Illustrated by three charts. Chart 1 is a type curve which shows the present relative position of electric refrigerator, oil burner, radio, and automobile markets. "It is based on the number in use, and is analogous to annual automobile registration figures. The probable degree of saturation (per capita use) is determined by the use of an integrated probability curve as the trend and significant changes of curvature correspond to relative changes of curvature in the S-curve . . . "Chart 2 illustrates the changes that have taken place in the character of the buying and the effect on production. It is based on smoothed three-year moving averages of annual figures for domestic passenger automobiles." Chart 3 shows consumption trend, marginal utility, and marginal demand price of reclaimed rubber and price trend of crude rubber, 1919–1928.

HEFLEBOWER, R. B. The use of statistics in verifying and clarifying economic theory. *Pacific Coast Economic Conference, Proceedings*, 10th, 1931, pp. 1–13.—The statistical analysis of demand, pp. 2–5. The statistical analysis of supply, pp. 5–7. Accompanied by five charts which show the demand curve for lamb in the wholesale market, 1907–1923; retail demand situation for potatoes, 1921–1929 (Chicago); receipt of potatoes at Chicago and prices of Wisconsin

round whites at Chicago preceding week, 1929–30 season; changes in cotton acreage in the United States in response to variations of the preceding year's price from average, 1903–1926 and of potatoes, 1921–1929; net effect of December corn-hog ratio upon receipts of hogs at Chicago, nine to fifteen months later.

HOPKINS, J. A., Jr. The forecasting of economic phenomena. *Iowa State Col. Jour. Sci.* **2** (4): 263–288. July 1928.—Literature cited, p. 288. Subtopics are as follows: Nature of the economic problem; The time factor in economic life; The development of forecasting methods; The use of correlation methods; Curvilinear correlation methods; Forecasting by use of a demand curve; Forecasting by analysis of the economic situation; Composite method; An example of price forecasting (cattle); Application and interpretation of forecasts; Permanence of price relationships; Forecasts as centers of a belt of probability.

HOTELLING, HAROLD. Edgeworth's taxation paradox and the nature of demand and supply functions. *Jour. Polit. Econ.* **40** (5): 577–616. Oct. 1932.

IVERSON, CARL. Efterspørgselens elasticitet. [Elasticity of demand.] *Nationaløkonomisk Tidsskrift* **68** (4): 313–341. 1930.—"A survey of the theoretical and practical difficulties which arise when one attempts to work with the factor of elasticity. The discussion which is amply supported by citations, examines the concept of elasticity itself, and the possibility of arriving at a concrete demand elasticity for certain commodities on the basis of statistics."—*Social Sci. Abs.*, v. **3**, Dec. 1931, item 18863.

JENKIN, FLEEMING. The graphic representation of the laws of supply and demand, and other essays on political economy. 154 pp. London, 1931. (London School Econ. and Polit. Sci. . . . No. 9 in Ser. Reprints of Scarce Tracts in Econ. and Polit. Science.)—Vol. 2 of his Papers, Literary, Scientific, etc., London, 1887, of which a reproduction of the title-page is included. The graphic representation of the laws of supply and demand, and their application to labour, pp. 76–106. From Recess Studies. Edited by Sir Alexander Grant. Edinburgh, 1870.

KUZNETS, S. S. Seasonal variations in industry and trade. 455 pp. New York, National Bureau of Economic Research, 1933.—In three parts: Part I. The economic problem of seasonal variations and their statistical measurement; Part II. Average seasonal variations in selected groups in industry and trade; Part III. The variability of seasonal movements. The following appendices are also given: I. Indexes of seasonal variations in the flow and stocks of commodities and in other aspects of economic activity in the United States; II. The sources of seasonal indexes used in the study but not included in appendix I; III. The lacunae in statistical data revealed by the study.

KUZNETS, S. S. Secular movements in production and prices: their nature and their bearing upon cyclical fluctuations. 536 pp. Boston and New York, Houghton Mifflin Company, 1930.—Reviewed by Dorothea D. Kittredge in *Jour. Farm Econ.* **13** (1): 177–179. Jan. 1931. Reviewed by Robert Weidenhammer in *Jour. Business Univ. Chicago* **4** (1): 96–97. Jan. 1931. "The present investigation attempts to inquire into the nature of the long-time movements in production and prices, as well as into their influence on cyclical fluctuations. It attempts to establish some uniformly observed characteristics of the so-called secular movements, and in the process to draw analytical distinctions between different groups of these movements. The latter are studied as they appear in the available statistical series of various countries. The aim is to formulate these movements in such a way as not only to reveal their common characteristic but also to facilitate measurements of the relation between cyclical oscillations and the underlying secular changes. A definite hypothesis concerning characteristics of industrial growth forms the basis of our statistical description of long-time movements. This hypothesis is derived from a preliminary observation of the wide field of statistical evidence and from material furnished by the chronicles of specific industries. The exact formulation of this tentative generalization and the historic-statistical material on which it rests are presented in Chapter I. The conclusions of the first chapter furnish a definite approach to the methods

of statistical analysis which can be applied to the body of available material. A consideration, however, of some of the general principles of statistical description of long-time movements makes clearer the exact type of analysis to pursue. This consideration forms the contents of Chapter II. Chapter III presents the body of statistical data drawn from five countries: United States, Great Britain, Belgium, Germany, and France. Altogether about 60 series of industrial output and 35 series of prices are analyzed. The secular movements are segregated from the cyclical oscillations and are analyzed into two distinct groups of changes, the primary and the secondary. Each pair of production and price series is accompanied by brief comments drawn from the history of the particular industry whose output and prices are described quantitatively. The measurable characteristics and the nature of the secondary secular movements . . . are discussed in Chapter IV. The discussion inquires into the causes of these secondary secular movements in production and prices, and deals with the question whether they can properly be called major cycles. The appendix to the chapter surveys the more important theories of major cycles set forth by various writers. In Chapter V, the influence of secular movements on the cyclical oscillations is studied statistically. The aim is to discover the existence or absence of correlation between the rapidity of secular growth and the amplitude of cyclical oscillations. The theoretical connection of these two measures of forces as revealed by contemporary business cycle theory is treated in Chapter VI. The concluding chapter deals briefly with the theoretical significance of the results of the investigation and attempts to evaluate them in the light of economic theory, both past and present."—Author's preface.

LEONTIEF, WASSILY. Pitfalls in the construction of demand and supply curves: a reply *Quart. Jour. Econ.* **48** (2): 355–361. Feb. 1934.—This is a reply to a criticism of a publication of Mr. Leontief made by Professor Ragnar Frisch in a pamphlet devoted to the description of the pitfalls in the statistical construction of demand and supply curves.

LEONTIEF, WASSILY. Studien über die elastizität des angebots. *Weltwirtschaftliches Archiv* **35** (1): 66–115. Jan. 1932.—"The 'total' elasticity of supply of any kind of goods depends upon the average elasticity of the ultimate costs and 'transformation elasticities' of the different successive stages of production. The latter represents the relation between changes in the marginal natural costs (measured 'in kind' by a quantity index) of production and the total output in a given stage of production. Two approaches to the statistical calculation of the 'transformation elasticities' are developed. A statistical analysis of the American iron market (1879–1915) shows how the significant changes in the supply schedules were resulting from widely heterogeneous developments in the 'partial' elasticities, i.e., cost and 'transformation' elasticities of the different stages of production." *Social Sci. Abs.*, v. **4**, May 1932, item 7586.

LEONTIEF, WASSILY. Ein versuch zur statistischen analyse von angebot und nachfrage. *Weltwirtschaftliches Archiv* **30** (1): 1–53. July 1929.—"This is an attempt to deduce statistically the demand and supply curves of commodities by a method which differs from that followed by American economists. Whereas the latter generally adjust their raw data for changes in population, in the purchasing power of money, and in other factors, and then apply the correlational calculus either directly to the adjusted data, or to trend ratios or link relatives, sometimes lagging one series behind the other, Leontief derives his curves by a single calculation . . . "—*Social Sci. Abs.*, v. **2**, Mar. 1930, item 2088.

LERNER, A. P. The diagrammatical representation of elasticity of demand. *Review of Economic Studies* **1** (1): 39–44. Oct. 1933.

MAJEROTTO, SERAFINO. Un nuovo metodo per la determinazione empirica delle curve di domanda e offerta. *Rivista Internazionale di Scienze Sociali e Discipline Ausiliarie* **3** (4): 528–539. July 1932.—Discussion of Jakob Marschak's Elastizität der Nachfrage, 1931.

MARSCHAK, JAKOB. Elastizität der nachfrage. Zur empirischen festellung relativer marktkonstanten durch beobachtung von haushalt, betrieb und markt.

143 pp. Tübingen, J. C. B. Mohr (P. Siebeck) 1931. (Beiträge zur ökonomischen theorie hrsg. von Emil Lederer und Joseph Schumpeter 2.)—Reviewed by Henry Schultz in *Weltwirtschaftliches Archiv;* Zeitschrift des Instituts für Weltwirtschaft und Seeverkehr und Seeverkehr an dor Universität Kiel **37** (1): 29–38. Jan. 1933. The first several paragraphs of this review by Schultz are devoted to a statement of the contributions of H. L. Moore (1914) and others to the problem of deriving the elasticity of demand from statistics. Then he states: "Very recently, however, several other attacks on the problem of deriving the elasticity of demand from statistics have come to the forefront. Though they differ among themselves, they have this characteristic in common: They are attempts to derive the demand curve and the elasticity of demand from income data or from family budget data rather than from time series of prices and quantities . . . The present work by Marschak is the first attempt at a systematic presentation and criticism of these writings and presents his own interesting and important attack on the same subject. An introductory part of 53 pages deals with economic equilibrium, with cost data and cost curves, and with the market data approach to the study of demand. This is followed by a theoretical and by a statistical exposition of his own method of deriving demand curves consisting respectively of 37 and 29 pages. The last part of the book, consisting of 24 pages, contains an exposition and comparison of the ideas of Pigou, Vinci, Tschayanow, Fisher, and Frisch." The "rationale" of Marschak's procedure is summarized.

MASCI, GUGLIELMO. Metodi statistici per lo studio dell'equilibrio general dinamico. [Statistical methods for the study of general dynamic equilibrium.] *Economia* 9 (4): 375–400. Apr. 1932.—"The dynamical development of the series of prices and of the quantities of consumption and production has been the object, in the last few years, of many inductive statistical analyses with the purpose of distinguishing in them the various kinds of fluctuations and of obtaining the empirical laws of demand and supply. Such researches have been made especially in the United States by H. L. Moore. He proceeds from the abstract equations of Walras, expressing the conditions of equilibrium of exchange and of production in a static market and on the basis of various hypotheses seeks to determine the real concrete form that such functions assume and to outline a system of dynamic equilibrium developing along the line of secular trend. But in reality Moore's method permits the determination, for example, of the variations of demand in function of price, not in function of time; he gives, therefore, the static law of demand, not the dynamic; only indirectly can it furnish some notion of dynamics as long as several successive periods are observed. But even apart from this and from the arbitrary character inherent in the interpolative method of determining the trend and in the choice of the form of the function representing the law of demand, Moore's system does not succeed in giving any explanation of the cause of dynamic factors acting in the time and the place studied. Still less can it give a valid universal explanation of dynamic fluctuations, an outline of the way according to which economic reality can develop. Yet this is the object of economics, an object which can be reached only with deductive reasoning. Thus, statistical researches like Moore's, even though very useful, cannot presume to be a substitute for abstract economic science."—*Social Sci. Abs.*, v. 4, Dec. 1932, item 20724.

MASCI, GUGLIELMO. Sulla determinazione statistica della curva di domanda. *Rivista Italiana di Statistica, Economia e Finanza* **4** (3): 476–514. Sept. 1932.—Discussion by F. Vinci and G. Masci, pp. 509–514.

MEHTA, J. K. The nature and intensity of demand. *Indian Jour. Econ.* **12** (4): 501–506. Apr. 1932.

MERIAM, R. S. Supply curves and maximum satisfaction. *Quart. Jour. Econ.* **42** (2): 169–198. Feb. 1928.

MILLER, H. E. Utility curves, total utility, and consumer's surplus. *Quart. Jour. Econ.* **41** (2): 292–316. Feb. 1927.

MOORE, H. L. Partial elasticity of demand. *Quart. Jour. Econ.* **40** (3): 393–401. May 1926.

MOORE, H. L. Synthetic economics. 186 pp. New York, The Macmillan Company, 1929.—Titles of chapters are Introduction; Fundamental notions; The law of demand; The law of supply; Moving equilibria; Economic oscillations; Conclusion. Reviewed by Mordecai Ezekiel in *Quart. Jour. Econ.* **44** (4): 663–679. Aug. 1930. Reviewed by P. G. Wright in *Jour. Polit. Econ.* **38** (3): 328–344. June 1930.

MORET, JACQUES, and FRISCH, RAGNAR. Méthodes nouvelles pour mesurer l'utilité marginale. *Revue d'Économie Politique* **46** (1): 1–28. Jan.–Feb. 1932.—This article is in two parts. The second part is Frisch's discussion of J. Marschak's paper on the statistical determination of supply and demand curves given at the September 1931 meeting of the Econometric Society at Lausanne, together with a note added after the conference. In this discussion the validity of the formula by which Marschak derives his demand curve from a budgetary curve is questioned.

MORETTI, VINCENZO. Sopra alcuni problemi di dinamica economica. *Giornale degli Economisti e Rivista di Statistica* **44** (7): 449–488. July 1929.—"This article is a defense of traditional doctrine against new theories advanced by American economists which are founded on statistical observations . . . In the second part of the article the author discusses the problem of demand and of the equilibrium of prices between demand and supply. The solution given by the American school headed by Moore is explained and criticized. The acceptance of the new methods introduced by Moore and Mitchell would mean a complete annihilation of the static theory of demand and of all the traditional theories connected with it. Edgeworth, Marshall, and Pareto, have doubted the possibility of building demand curves by statistics. It is shown that such curves cannot be constructed. The author denies the existence of a problem of statistical verification of the law of demand. He recognizes the existence of statistical curves of demand for a certain quantity of goods but he is sceptical as to their meaning and content and as to the possibility of applying them to the forecasting of future trends."—*Social Sci. Abs.*, v. **2**, Feb. 1930, item 1449.

NATIONAL BUREAU OF ECONOMIC RESEARCH, INC. Report of the Director of research for the year 1933. 32 pp. New York, 1934.—"Another of our special studies in this field is being conducted by Dr. Wassily Leontief who joined the staff of Harvard University after spending the year 1931–32 with us as Research Associate. Dr. Leontief is making a statistical analysis of shifts in demand for and supply of commodities during business cycles. He makes the following report upon his progress: 'The first part of the study contains a discussion of fundamental concepts, especially the relation between the (Marshallian) supply and demand curves, and changes in the general equilibrium, the whole system. The next part will contain a detailed description of the statistical technique which is applied in the third part to the empirical analysis of some twenty-five American commodity markets over a period of from seven to thirteen cyclical waves. In the concluding section, I shall sum up the results in a comparative study of cyclical shifts.' "—pp. 12–13.

NICHOL, A. J. Measures of average elasticity of demand. *Jour. Polit. Econ.* **39** (2): 249–255. Apr. 1931.—A further note on this subject by Nichol is given in *Jour. Polit. Econ.* **39** (5): 658–661. Oct. 1931. "The measure of elasticity of demand developed by Alfred Marshall may be applied only to individual points on a demand curve. The measures of elasticity suggested in this paper represent an extension of Marshall's idea to any given arc of a demand curve. The measures of arc elasticity depend upon (1) the kind of average employed and (2) the system of weighting. Several different measures are given and some of their properties are discussed."—*Social Sci. Abs.*, v. **3**, 1931, item 18289.

PALGRAVE, SIR ROBERT. Palgrave's dictionary of political economy; ed. by Henry Higgs. 3v. London, Macmillan and Company, Limited, 1925–1926.—Demand, by J. N. K., v. **1**, pp. 539–542; Demand curves, by F. Y. E., v. **1**, pp. 542–544; Supply, by W. E. J., v. **3**, pp. 488–490; Supply and demand, by W. E. J., v. **3**, pp. 490–496; Supply-curves, by F. Y. E., v. **3**, pp. 497–498.

PETRICKSO, MIKOLAUS. A theory of prices. *Jour. Polit. Econ.* **40** (6): 808–813. Dec. 1932.

PIGOU, A. C. An analysis of supply. *Econ. Jour.* **38** (150): 238–257. June 1928.

PIGOU, A. C. Demand supply equations. In Pigou, A. C., and Robertson, D. H. *Economic Essays and Addresses*, pp. 84–94. London, P. S. King & Son, Ltd., 1931.

PIGOU, A. C. The elasticity of demand from family budgets. *Quart. Jour. Econ.* **47** (2): 342. Feb. 1933.—The statement given is as follows: "In an article under this title by Mr. Waugh in the November issue of this *Journal* there is an unintentional misrepresentation of my discussion of this matter in Appendix II to my *Economics of Welfare*. According to Mr. Waugh, I obtain a 'budgetary elasticity' for particular commodities (as defined by him on p. 135) and then mistakenly treat this as equivalent to elasticity in the Marshallian sense. In fact, I am concerned solely with elasticity in the Marshallian sense, and I expressly state that it is not possible to derive this elasticity for particular commodities from family budgets. My analysis is directed to show that the comparative, not the absolute, elasticities of demand for different commodities can be so derived."

PIGOU, A. C. Industrial fluctuations. 397 pp. London, Macmillan and Co., Limited, 1927.—The following is quoted from the author's preface: "In the first edition of my *Economics of Welfare* (1920), Part VI, containing 112 pages, was entitled 'The Variability of the National Dividend.' In the second edition (1924) this part was omitted because I was anxious to undertake a more comprehensive study of industrial fluctuations. The present volume is the result of this study. . . . The conditions prevailing in the great post-war boom and subsequent depression have been so abnormal that I have not examined them here . . ." Consult the index under headings—supply, demand, prices, etc.

PIGOU, A. C. The statistical derivation of demand curves. *Econ. Jour.* **40** (159): 384–400. Sept. 1930.—Reprinted in Pigou, A. C., and Robertson, D. H. *Economic Essays and Addresses*, pp. 62–83. London, P. S. King & Son. Ltd., 1931.

RICCI, UMBERTO. Ancora sull'offerta del risparmio. *Giornale degli Economisti e Rivista di Statistica* **42** (9): 482–504. Sept. 1927.—In two parts: Curve individuali; Curve collettive. Topic headings in part 1: L'equilibrio a interesse inesistente; Curva individuale di offerta; Curva individuale di domanda; Completamento della curva individuale di offerta; Completamento della curva individuale di domanda; Tracciamento di una curva individuale di domanda e offerta; Riassunto. Topic headings in part 2: Curva collettiva unica di domanda e offerta; Curve collettive unica di domanda e offerta; Curve collettive separate di domanda e di offerta; Equilibrio tra domanda totale e offerta totale; Necessità sociale dell' interesse.

RICCI, UMBERTO. Courbes de la demande et courbes de la dépense. *L'Egypte Contemporaine* **22** (129): 556–588. May 1931.—"Mathematical discussion of demand and supply, contrasting the methods and views of Cournot and Marshall."—*Social Sci. Abs.*, v. **4** Jan. 1932, item 646.

RICCI, UMBERTO. Elasticità dei bisogni della domanda e dell'offerta. [Elasticity of requirements of demand and supply.] *Giornale degli Economisti e Rivista di Statistica* 65, yr. **35** (8, 10): 413–431, 509–531. Aug., Oct. 1924.—Translated titles of chapters are: Concept of elasticity in diminishing curves (Definitions of elasticity of demand, elasticity of requirements, and price flexibility); Static and dynamic demand curves; Diminishing elasticity curves; Increasing elasticity curves; Constant elasticity curves; Points of elasticity equal to 1; Points of elasticity above or below 1; Elasticity in increasing curves; Average elasticity.

RICCI, UMBERTO. Klassifikation der nachfragekurven auf grund des elastizitätsbegriffes *Archiv für Sozialwissenschaft und Sozialpolitik* **66** (1): 36–61. Aug. 1931.—Classification of demand curves according to their elasticity. Abstract in *Social Sci. Abs.*, v. **4**, May 1932, item 7588.

RICCI, UMBERTO. La loi de la demande individuelle et la rente de consommateur. *Revue d'Économie Politique* **40** (1): 5–24. Jan.–Feb. 1933.—Errata of this article in *Revue d'Économie Politique* **40** (3): 944. May–June, 1933.

RICCI, UMBERTO. On the demand for rival (or substitute) commodities. ECONOMETRICA **1** (2): 181–189. Apr. 1933.

RICCI, UMBERTO. The psychological foundation of the law of demand. *Jour. Polit. Econ.* **40** (2): 145–185. Apr. 1932.—Translated from the Italian by Henry Julian Wadleigh. "An algebraic and diagrammatic analysis of the relations between utility curves and demand curves. . . . "—*Social Sci. Abs.*, v. **4**, Aug. 1932, item 13185.

RICCI, UMBERTO. Può una curva di domanda esser crescente? *Giornale degli Economisti e Rivista di Statistica* **47** (4): 197–240. Apr. 1932.—"The curve of demand has an increasing segment and a decreasing segment; the increasing segment is one of unstable equilibrium. The true curve of demand, economically valid, commences from the point where the curve begins to decrease. The true curve of demand is generated from the curve of utility. Errors have arisen from (1) confusing increasing demand with increasing cost; (2) confusing the case of constant price for the successive units of a good with the case of variable price proportional to the utility of each unit; (3) confusion of the simple curve of demand with the dynamic curve of utility, or the static curve with the historical curve of demand."—*Social Sci. Abs.*, v. **4**, Dec. 1932, item 20732.

RICCI, UMBERTO. Die statistischen gesetze des gleichgewichtes nach Henry Schultz. *Zeitschrift für Nationalökonomie* **2** (3): 305–333. Jan. 15, 1931.—In five parts: Das gesetz der nachfrage laut Professor Schultz; Erörterung und kritik des gesetzes der nachfrage; Das angebot an zucker in den vereinigten staaten; Weltangebot und weltnachfrage nach zucker; Rückwirkungen eines zolles.

RICCI, UMBERTO. Die "synthetischeokonomie" von Henry Ludwell Moore. *Zeitschrift für Nationalökonomie* **1** (5): 649–688. Apr. 30, 1930.—Abstracted in *Social Sci. Abs.*, v. **3**, Apr. 1931, item 5539.

ROBBINS, LIONEL. The economic works of Philip Wicksteed. *Economica*, no. 30, Nov. 1930, pp. 245–258.—The object of this article as stated by the writer is "to give some account of Wicksteed's main works in the field of economics" and "to attempt a brief estimate of the significance of his various contributions." The following is quoted from p. 255: "A second contribution which must always be associated with his name is his famous demonstration of the reversibility of the market supply curve. The general proposition that the reservation prices of sellers are in the ultimate analysis demands, was one which he continually reiterated with varying shades of emphasis. 'What about the supply curve that usually figures as a determinant of price, co-ordinate with the demand curve?' he asked in his address as President of Section E of the British Association in 1913—an address which epitomizes many of his most characteristic doctrines. 'I say it boldly and baldly: there is no such thing. When we are speaking of a marketable commodity, what is usually called the supply curve is in reality the demand curve of those who possess the commodity, for its shows the exact place which every successive unit of the commodity holds in their relative scale of estimation. The so-called supply curve therefore, is simply a part of the total demand curve. . . . The separating out of this portion of the demand curve and reversing it in the diagram is a process which has its meaning and its legitimate function . . . but is wholly irrelevant to the determination of the price.' It is safe to say that no one who has followed through his beautiful diagrammatic analysis of this proposition, and realized its wider implication that all psychological variables can be exhibited as phenomena of demand acting on fixed stocks or products or factors of time, will deny that the whole of the analysis of economic equilibrium has received thereby a transforming elucidation."

ROBBINS, LIONEL. An essay on the nature & significance of economic science. 141 pp. London, Macmillan & Co., Limited, 1932.—The writer in his preface states that "the purpose of this essay is twofold. In the first place, it seeks to arrive at precise notions concerning the subject-matter of Economic

Science and the nature of the generalisations of which Economic Science consists. Secondly it attempts to explain the limitations and the significance of these generalisations, both as a guide to the interpretation of reality and as a basis for political practice." Chapter V is on Economic Generalisations and Reality. Pages 98–101 of this chapter are on Statistical Laws of Supply and Demand.

ROBBINS, LIONEL. On the elasticity of demand for income in terms of effort. *Economica*, no. 29, June, 1930, pp. 123–129.

ROOS, C. F. A dynamical theory of economics. *Jour. Polit. Econ.* **35** (5): 632–656. Oct. 1927.—In four parts: Introduction; Differential equations relating rates of production and prices; Functional equations relating rates of production and prices; A dynamical theory of economic equilibrium.

ROOS, C. F. A mathematical theory of price and production fluctuations and economic crises. *Jour. Polit. Econ.* **38** (5): 501–522. Oct. 1930.—The writer states the problem as follows: "Various papers and treatises have appeared exhibiting curves and statistics indicating that prices periodically oscillate about a given price trend. Other investigations have indicated a series of plus and minus deviations from a given datum line which pass through definite phases but do not recur with strict periodicity. Still other papers have indicated plus and minus deviations which do not reveal definite characteristic phases and do not recur. In spite of these researches, no successful attempt to relate the theory of business cycles with economic theory has been made. It is this problem which I attack here."

ROOS, C. F. Theoretical studies of demand. ECONOMETRICA **2** (1): 73–90. Jan. 1934.—Titles of the seven parts of this article are as follows: The rôle of mathematics in economics; Factors influencing demand; Dependence of demand upon past prices; Price forecasting and speculation; Time lags; The elasticity of demand; Conclusion.

ROSSI, LIONELLO. Del concetto di elasticità in economia [**The concept of elasticity in economics.**] *Giornale degli Economisti e Rivista di Statistica* **47** (1): 18–26. Jan. 1932.—"(Discussion of the definitions of Marshall, Moore, Ragnar Frisch, Barone, and others.) The ratio of the elasticities of two variables with respect to a third is equal to the elasticity of the first with respect to the second. The product of the elasticity of a variable with respect to a second by the elasticity of the second with respect to a third is equal to the elasticity of the first with respect to the third. The elasticity of demand (with respect to prices) is equal to the reciprocal of the elasticity of marginal utility (with respect to the quantity of the good)."—*Social Sci. Abs.*, v. **4**, Sept. 1932, item 14895.

ROSSI, LIONELLO. L'elasticità della domanda e la traslazi one dell'imposta in regime di monopolio. *Giornale degli Economisti e Rivista di Statistica* **47** (8): 600–607. Aug. 1932.

ROY, RENÉ. La demande dans ses rapports avec la répartition des revenus. *Metron* **8** (3): 101–153. 1930.—A paper with the same title presented at the September 1931 meeting of the Econometric Society at Lausanne, was published in ECONOMETRICA **1** (3): 265–273. July 1933.

ROY, RENÉ. Les lois de la demande. *Revue d'Économie Politique* **45** (4): 1190–1218. July–Aug. 1931.

SALZ, ARTHUR. Der begriff der "elastizität in der theoretischen nationalökonomie. *Archiv für Sozialwissenschaft und Sozialpolitik* **57** (2): 336–391. Apr. 1927.

SCHMIDT, ROBERT. Die pragnanz der elastizitatskoeffizienten. *Weltwirtschaftliches Archiv* **32** (1): 264–273. July 1930.—The statistical significance of coefficients of elasticity of demand and supply. Abstracted in *Social Sci. Abs.*, v. **3**, 1931, item 3414.

SCHNEIDER, ERICH. Kostenanalyse als grundlage einer statistischen ermittlung von nachfragekurven. *Archiv für Sozialwissenschaft und Sozialpolitik*

66 (3): 585–605. 1931.—"Assuming that the demand curve for a commodity is one of uniform elasticity, or uniform flexibility of price as defined by Moore, and assuming that the commodity is produced by a monopoly which at any given time correctly adjusts its production to sales at a price which gives the maximum net profit, then the equation for the demand curve can be calculated by applying reasoning of the type used by Cournot . . . "—*Social Sci. Abs.*, v. 4, July 1932, item 11374.

SCHNEIDER, ERICH. Kostentheoretisches zum monopolproblem. [Cost theory and the monopoly problem] *Zeitschrift für Nationalökonomie* 3 (2): 185–211. Dec. 1931.—"The well-known Marshallian use of constant outlay curves in determining the monopoly price is not appropriate when we wish to analyze the relations between price and cost of production of a monopolistic enterprise from the marginal point of view. A new method more suitable for this purpose is based on the fundamental concept of 'marginal receipts' and the fact that the monopoly point is determined by equality of marginal receipts and marginal cost of production. The latter condition leads to a series of interesting conclusions in this chapter. In section 2 the variations of marginal receipts brought about by variations of demand curves are discussed. Section 3 contains a detailed analysis of the concept of marginal cost of production when the prices of the factor of production are constant or when the scale of production varies and the enterprise reacts on variations in the scale of production by partial or total adaptation. In the last chapter the variations of monopoly price brought about by variation of the demand curve are studied."—*Social Sci. Abs.*, v. 4, Apr. 1932, item 5598.

SCHNEIDER, ERICH. Über den einfluss von änderungen der nachfrage auf die monopolpreisbildung. *Archiv für Sozialwissenschaft und Sozialpolitik* 64 (2): 281–315. Oct. 1930.—"Cournot's theory of monopoly assumes static demand and cost curves for a definite commodity. Since the structures of the demand and cost curves are subject to continual fluctuation and change, the monopoly price of a commodity affected by these variations is likewise subject to continual change. The author examines the relationship of the Cournot theory of monopoly to what he characterizes as the Marshallian concept of elasticity of demand, and also the theoretical possibilities and probabilities with respect to the shifting and changes in form of demand curves in time. He then treats of these variations in the demand-curve as related to the problem of monopoly. The assumptions made in the course of his argument with respect to the possible changes of a demand curve in time are found to be not inconsistent with the results of certain empirical-statistical investigations—such as those of Schultz." —*Social Sci. Abs.*, v. 3, Oct. 1931, item 15565.

SCHNEIDER, ERICH. Ueber die nachfrage nach produktions—mitteln und ihre elastizität. *Jahrbücher für Nationalökonomie und Statistik* bd. 137 (III folge. 82 bd.), hft. 6, Dec. 1932, pp. 801–814.—The demand for production goods and its elasticity.

SCHULTZ, HENRY. A comparison of elasticities of demand obtained by different methods. ECONOMETRICA 1 (3): 274–308. July 1933.—The writer has been working off and on for several years on a study of the demands for various goods in so far as they can be determined from market data, and in this connection he has had occasion "to work with time series of prices and quantities of varying degrees of reliability and to work with different methods of procedure. This paper is essentially a summary and a comparison of some of the results that have been obtained by the various methods in so far as they relate to the elasticity of demand. They invite comparison with the results that have been, or that can be derived by the budget data approach." Commodities whose elasticities of demand are compared and for which the writer has determined "not only the shape of the demand curve, but also its rates of shift from 1875 to 1929" are corn, hay, wheat, sugar, potatoes, oats, barley, rye, and buckwheat.

SCHULTZ, HENRY. Henry L. Moore's contribution to the statistical law of demand. Analysis no. 46 in *Methods in Social Science*, edited by Stuart A. Rice, pp. 645–661. Chicago, University of Chicago Press, 1931.

SCHULTZ, HENRY. Interrelations of demand. *Jour. Polit. Econ.* 41 (4): 468–512. Aug. 1933.—"This paper was read before the joint meeting of the

American Statistical Association with the Econometric Society at Cincinnati, Ohio, December 30, 1932." The following is quoted from the writer's introduction: "In two earlier papers I measured the elasticities of demand and the rates of shift of the demand curves for selected commodities, and compared the elasticities of demand obtained by different methods. In both studies the commodities considered were corn, hay, wheat, sugar, potatoes, oats, barley, rye, and buckwheat; the data used were the per capita consumptions and the 'real' (i.e. deflated) prices of these commodities; and the periods covered were three: (I) 1875–95, (II) 1896–1914, and (III) 1915–29 (excluding 1917–21) . . . No account was taken of the prices of related goods, and no analysis was made of the interrelations of the demands for the various commodities. "The main object of the present paper is to extend the analysis of demand into the field of related goods. More specifically, the objectives are: (1) to compare the theoretical properties of demand functions for completing and competing goods; (2) to analyze the concrete statistical demand functions of a group of related commodities with the view of determining which of them are completing and which are competing; (3) to see whether and to what extent the theoretical relations are reflected in the observations; (4) to call attention to certain new problems in statistical methodology; and (5) to indicate the practical significance of studies in the interrelations of demand, especially their bearing on the problem of taxation."

SCHULTZ, HENRY. Marginal productivity and the general pricing process. *Jour. Polit. Econ.* **37** (5): 505–551. Oct. 1929.—"It is the object of this paper (1) to deduce the theorems of the marginal-productivity theory in such a way as to lay bare their underlying assumptions; (2) to show the relation of marginal productivity to the general pricing process; (3) to suggest a statement of the relation between cost and price which appears to be more satisfactory than the current theory; and (4) to consider some aspects of the problem of the statistical measurement of the productivities of labor and capital which have not as yet been noticed."—pp. 505–506.

SCHULTZ, HENRY. The shifting demand for selected agricultural commodities, 1875–1929. *Jour. Farm Econ.* **14** (2): 201–227. Apr. 1932.—Summarized in Conference on Economic Policy for American Agriculture at the University of Chicago, September 7, 8, and 9, 1931, pp. 70–71. "Approximate statistical significance is given to the concepts elasticity of demand and shift of the demand curve. The per capita demand curves for the nine farm crops studied have either ceased to shift upward or have begun to shift downward. The only upward shift in demand to be expected in the near future is that due to the normal increase in population. But the rate of increase of our population has been decreasing, and population is rapidly approaching its upper limit. (Two tables including equations of demand for the nine farm crops for three major periods; 8 charts)."—*Social Sci. Abs.*, v. **4**, Oct. 1932, item 16978. The commodities studied are corn, wheat, potatoes, barley, rye, buckwheat, hay, oats, and sugar.

SCHULTZ, HENRY. Der sinn der statistischen nachfragekurven. *Frankfurter Gesellschaft für Konjunkturforschung. Veröffentlichungen*, hft. 10, 99 pp. Bonn, K. Schroeder, 1930.—Bibliography, pp. 98–99. Translated from the English by Hans Staehle. Photostatic copy (in U.S.D.A., B.A.E. Library) of original manuscript from which German translation was made reviewed by L. H. Bean in *Jour. Amer. Statis. Assoc.* **26** (n.s. 176): 481–484. Dec. 1931.—Abstract by Henry Schultz in *Social Sci. Abs.*, v. **4**, 1932, item 17634.

SCHULTZ, HENRY. The standard error of the coefficient of elasticity of demand. *Jour. Amer. Statis. Assoc.* **28** (181): 64–69. Mar. 1933.

SHEPHERD, G. S. The burden of increased costs of distribution. *Jour. Farm Econ.* **14** (4): 650–661. Oct. 1932.—"An examination of the shifts which agricultural supply schedules undergo when the costs of distribution are increased, when business is depressed, or when the general price level is lowered." —Wells, O. V. Farmers' response to price, item 13.

SHEPHERD, G. S. Supply and production, demand and consumption. *Jour. Farm. Econ.* **13** (4): 639–642. Oct. 1931.

SHEPHERD, GEOFFREY. Vertical and horizontal shifts in demand curves. *Jour. Farm Econ.* **15** (4): 723–729. Oct. 1933.—In an article in the *Journal of Farm Economics* for July, 1933, Mr. F. L. Thomsen questioned the validity of the distinction between vertical and horizontal shifts in the position of the demand curve which Mr. Shepherd had made in an article in the *Journal of Farm Economics* for October, 1931. The present "note" by Mr. Shepherd provides a fuller treatment of this distinction.

SLICHTER, S. H. Modern economic society. 909 pp. New York, H. Holt and Company [1931].—References at end of chapters. See particularly chapter XIII, The Stability of Prices under Competition, pp. 282–301. This chapter includes the following parts: Some factors influencing the stability of prices; The elasticity of supply and of demand; The effect of the elasticity of demand and of supply upon price movements; The determinants of the elasticity of demand; The determinants of the elasticity of supply; The effect of fixed costs upon the elasticity of supply; The effect of price movements upon the level of supply and demand.

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CIRCULATION PLANNING

By RAGNAR FRISCH

PART III. MATHEMATICAL APPENDIX

In the following sections, I shall give certain mathematical formulae on which part of the argument in the preceding Sections is built. Some of these formulae are not new, but are here derived in a manner which, in my opinion, is simpler and more systematic than the ones usually found in the literature: this applies in particular to the formulae of Section 23. Other formulae here given I do not recall to have seen before, for instance, those of Section 21.

21. Lemmas on Matrices Consisting of Non-Negative Elements

Consider a square matrix

$$(21.1) \qquad (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \cdot & \cdot \cdot \cdot \cdot \\ a_{n1} & a_{n2} \cdots a_{nn} \end{pmatrix}$$

whose elements are non-negative numbers. A request matrix in circulation planning has this property.

The operation which consists in extending a summation to one (or both) of the subscripts on a_{ij} we shall denote simply by replacing the subscript in question by a zero. Thus the sums of rows and columns respectively are denoted.

$$(21.2) \qquad \begin{aligned} a_{i0} &= \sum_k a_{ik} = a_{i1} + \cdots + a_{in} \\ a_{0j} &= \sum_k a_{kj} = a_{1j} + \cdots + a_{nj}. \end{aligned}$$

If the matrix (21.1) is such that

$$(21.3) \qquad a_{i0} = a_{0i} \text{ for all } i = 1, 2 \cdots n$$

we shall say that it has complete row-column equality. A request matrix will in general not have this property.

Let us consider the matrix obtained from (21.1) by subtracting from each diagonal element the sum of all the elements in the corresponding row, i.e. the matrix

$$(21.4) \qquad \begin{pmatrix} a_{11} - a_{10} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - a_{20} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} - a_{n0} \end{pmatrix}.$$

We may generalize the matrix (21.4) by further subtracting from each diagonal element a set of non-negative quantities $t_1, \cdots, t_n$, so that we obtain the matrix

$$(21.5)\qquad \begin{pmatrix} a_{11} - a_{10} - t_1 & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - a_{20} - t_2 & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - a_{n0} - t_n \end{pmatrix}.$$

For the following discussion it is essential that all the original elements and all the quantities t_i be non-negative. It should also be noted that the quantities a_{i0} denote the *complete* sum of the i-th row in the n rowed matrix (21.1). This is essential as we shall later consider certain minors of the above matrices.

The determinant value of (21.4) is equal to zero. This is seen simply by adding to the elements in any given column of (21.4) the elements of all the other columns; this will produce a column consisting exclusively of zeros.

For the more general determinant defined by (21.5) we have the following:

LEMMA I.—*If all the original elements a_{ij} and all the magnitudes t_i are non-negative, the determinant value of (21.5) has the sign $(-)^n$, in other words this determinant is non-negative if n is even and non-positive if n is odd.*

To prove this, let us denote the determinant in question by T, further let $T_{)i(}$ be the $(n-1)$-rowed determinant obtained by omitting the row No. i and the column No. i from T (but all the a_{k0} denoting still the sums of the rows in the complete determinant 21.1), further let $T_{)ij(}$ be the $(n-2)$-rowed determinant obtained by omitting the rows Nos. i and j and the columns Nos. i and j. And so on. Memotechnically we may interpret the inverted parenthesis) (to mean "exclusion of." By convention we shall put any expression of the form $T_{)\alpha\beta\ldots(}$ equal to zero whenever at least two of the subscripts are equal; and if *all* the n subscripts occur in the omission parenthesis we put $T_{)12\ldots n(}=1$. Finally let S, $S_{)i(}$, etc. denote the corresponding determinant and minors obtained from (21.4). Thus S, $S_{)i(}$, etc. depend only on the elements of the original matrix (21.1), while T, $T_{)i(}$, etc. depend also on $t_1 \ldots t_n$.

This notation being adopted, we see that if T is considered as a function of the n variables $t_1 \ldots t_n$, the partial derivatives of T are

$$(21.6)\qquad \frac{\partial T}{\partial t_i} = -T_{)i(} \qquad \frac{\partial^2 T}{\partial t_i \partial t_j} = T_{)ij(} \qquad \frac{\partial^3 T}{\partial t_i \partial t_j \partial t_k} = -T_{)ijk(}, \text{ etc.}$$

This shows that we have the Taylor expansion

$$T = S - \sum_i t_i S_{)i(} + \sum_{i<j} t_i t_j S_{)ij(} - \sum_{i<j<k} t_i t_j t_k S_{)ijk(} + \cdots + (-)^n t_1 t_2 \cdots t_n \tag{21.7}$$

where i runs through all the numbers $1, 2, \cdots, n$, $i<j$ runs through combinations without repetition of the two affixes (i, j) picked in the set $1, 2, \cdots, n$, etc.

The first term of the expansion (21.7), namely the determinant S is, as we have seen, equal to zero. This being so, the expansion (21.7) furnishes an easy means of proving Lemma I by complete induction. The lemma is obviously correct for $n=1$ because we have $a_{11}-a_{10}-t_1 = -t_1 =$ non-positive if $t_1 \geqq 0$. Therefore, let us assume that the lemma is correct for all orders up to and including $n-1$. Consider then the determinant $S_{)i(}$. In this determinant we separate from a_{10} the term a_{1i} which we denote for a moment t_1; from a_{20} we separate a_{2i} which we denote t_2, etc. Writing $S_{)i(}$ in this form we see that $S_{)i(}$ is an $(n-1)$-rowed determinant of the form (21.5). Indeed what is left of a_{10} when the term a_{1i} is taken out is just the sum of all the a's that occur in the first row in $S_{)i(}$, and similarly for the other rows. Consequently if Lemma I is correct for $n-1$, $S_{)i(}$ must have the sign $(-)^{n-1}$.

Next consider the determinant $S_{)ij(}$. From a_{10} in this determinant we take out $a_{1i}+a_{1j}$ and denote for a moment this binome t_1. From a_{20} we take out $a_{2i}+a_{2j}$ and denote it t_2, etc. This shows that $S_{)ij(}$ is an $(n-2)$-rowed determinant of the form (21.5). Hence it must have the sign $(-)^{n-2}$ provided Lemma I is correct up to $n-1$.

Quite generally any of the determinants $S_{)ij\ldots k(}$ must have the sign $(-)^{n-\nu}$ where ν is the number of subscripts in $S_{)ij\ldots k(}$. This shows that all the terms in the right member of (21.7) have the sign $(-)^n$, in other words, Lemma I is correct also for n.

LEMMA II.—*In the adjoint of the matrix (21.4) all the rows are equal, in other words, if b_{ik} are the elements of the adjoint, b_{ik} is independent of i.*

The element b_{ik} of the adjoint is defined as $(-)^{i+k}$ times the value of the $(n-1)$-rowed determinant obtained by crossing out the row k and the column i in the original matrix (note the interchange of rows and columns when we pass from the original matrix to the adjoint). For $n=3$ we have for instance

$$\begin{aligned} b_{11} &= (a_{22}-a_{20})(a_{33}-a_{30}) - a_{32}a_{23} &&= a_{21}a_{31} + a_{23}a_{31} + a_{21}a_{32} \\ b_{21} &= -(a_{33}-a_{30})a_{21} + a_{31}a_{23} &&= a_{21}a_{31} + a_{23}a_{31} + a_{21}a_{32}. \\ b_{31} &= a_{21}a_{32} - a_{31}(a_{22}-a_{20}) &&= a_{21}a_{31} + a_{23}a_{31} + a_{21}a_{32} \end{aligned} \tag{21.8}$$

This shows that for $n=3$ b_{i1} is independent of i.

For the general case Lemma II may be proved thus. Consider the determinant that defines b_{ik}. It contains all the columns of (21.4) except the column i. Let j be an arbitrary affix different from i, this means that the column j actually occurs in the determinant considered. This column j consists of the elements $a_{1j}, a_{2j}, \cdots,$) except $a_{kj}(\cdots a_{nj}$. To this column we add all the other columns. By this operation the column in question becomes $-a_{1i}, -a_{2i}, \cdots)$ except $-a_{ki}(\cdots, -a_{ni}$. The common factor -1 we take outside the determinant. The result of the operation is thus simply that in the column in question the elements have had their second subscript changed from j to i. We now *move* the column in question to the position it ought to have according to its new second subscript. By this movement the determinant is multiplied by $(-)^{i-j+1}$. This shows that the determinant that defines b_{ik} is equal to $(-)^{i-j}$ times the determinant that defines b_{jk}. The magnitude b_{ik} itself will consequently be equal to b_{jk}, in other words, b_{ik} is independent of i.

From the above proof follows that Lemma II applies, not only to matrices of the special type (21.4), *but more generally to all matrices where the sum of the elements in each row is zero.*

LEMMA III.—*All the n^2 elements in the adjoint of the matrix (21.4) have the same sign, namely $(-)^{n-1}$.*

This follows immediately from the Lemmas I and II. Indeed, from Lemma I follows that all the diagonal elements in the adjoint of (21.4) have the sign $(-)^{n-1}$. But on the other hand we know from Lemma II that all the rows of the adjoint are equal; consequently all the elements in any row must have the sign $(-)^{n-1}$. (21.8) furnishes an example for the case $n=3$.

The above propositions can of course easily be formulated in terms of columns instead of rows. Thus, for instance, if from the diagonal elements of the given matrix we subtract the *column* sums a_{0j} (instead of the row sums a_{i0}), the adjoint will have all its *columns* equal.

22. Solution of Certain Singular Systems of Linear Equations

Consider a system of linear equations of the form

$$\sum_{k=1}^{n} a_{ik}\xi_k = b_i \qquad (i = 1, 2 \cdots n) \tag{22.1}$$

where ξ_k are unknowns to be determined, and a_{ik} and b_i given constants (the a_{ik} are here taken quite generally as certain constants, not necessarily the elements of the request-matrix).

Suppose that the matrix of the coefficients a_{ik} is of rank $n-1$; the necessary and sufficient condition that the system has a solution is the

vanishing of all the n determinants obtained by replacing first the first column, then the second column, etc. in the determinant, a_{ik} by the numbers $b_1 \cdots b_n$. If this condition is fulfilled the solution is obtained by leaving out one equation and solving the remaining $(n-1)$ equation on the assumption that one of the quantities ξ is an arbitrary parameter. The solution thus obtained may be written out explicitly in the following form:

$$\hat{a}_{pq} \cdot \xi_i = \hat{a}_{iq} \cdot \xi_p + \sum_{k=1}^{n} \hat{a}_{ik}^{)qp(} \cdot b_k \tag{22.2}$$

where $\hat{a}_{pq}$ are the elements of the adjoint of the given n rowed matrix a_{ik}, and $\hat{a}_{ik}^{)qp(}$ are the elements of the adjoint of the $(n-1)$ rowed matrix obtained by leaving out the row q and the column p from a_{ik}. By convention $\hat{a}_{ik}^{)qp(}$ is interpreted as equal to zero whenever $i=p$ or $k=q$, or both. With this convention the formula (22.2) holds good if i, p and q independently of each are put equal to any of the numbers $1, 2, \cdots, n$.

If a definite p is selected, all the unknowns are by (22.2) expressed in terms of one of them, namely ξ_p, which may be given an arbitrary value. Instead of thus expressing all the unknowns in terms of one of them, it is for many purposes more convenient to express them in terms of some linear combination $\lambda_1\xi_1 + \cdots + \lambda_n\xi_n$ of the unknowns, $\lambda_1 \cdots \lambda_n$ being certain weights. This solution is immediately obtained by multiplying (22.2) with λ_p and performing a summation over p. The advantage of writing the solution in the form (22.2) is just that we may thus transform the solution to the particular form appropriate for any given case. For instance, if we want to express all the unknowns in terms of their sum $X = x_1 + \cdots + x_n$, we get from (22.2)

$$\hat{a}_{0q} \cdot \xi_i = \hat{a}_{iq} X + \sum_{k=0}^{n} \hat{a}_{ik}^{)q0(} \cdot b_k \tag{22.3}$$

where

$$\hat{a}_{0q} = \sum_{k=1}^{n} \hat{a}_{kq} \qquad \hat{a}_{ik}^{)q0(} = \sum_{h=1}^{n} \hat{a}_{ik}^{)qh(}. \tag{22.4}$$

In this formula q is still arbitrary; if we want to, we may make the solution symmetric also in this affix by extending to (22.3) a summation over q; if desired we could also apply a system of weights depending on q.

The above formulae show that the solutions may be expressed as *linear forms* in the quantities that occur in the second member of the given equations, plus a term involving an arbitrary parameter. If the number of variables is great, say 12 or more, the determination of the

coefficients in these linear form involves so much computation that the work becomes virtually prohibitive. We need, therefore, some form of approximation method.

In discussing this problem we shall use a notation which is especially appropriate for the application which we want later to make to the request-matrix. Let us consider the system of equations in the form

$$\sum_{k=1}^{n} c_k(A_{ki} - e_{ki}) = s_i. \tag{22.5}$$

Here c_k are the unknowns to be determined, A_{ki} is any matrix such that $(A_{ki}-e_{ki})$ is of rank $n-1$, and s_i are given constants; e_{ki} are the unit numbers. The difference between the systems (22.1) and (22.5) is thus only formal, (22.5) is just as general as (22.1).

We shall solve (22.5) by an iteration method. This method will lead to a solution that may finally be written out in explicit form by means of a certain matrix whose elements are defined by infinite series.

We first write equation (22.5) in the form

$$\sum_k c_k A_{ki} = c_i + s_i. \tag{22.6}$$

In this equation we replace for a moment i by x, postmultiply by A_{xi} and perform a summation over x. This gives.

$$\sum_k c_k \sum_x A_{kx} A_{xi} = \sum_x c_x A_{xi} + \sum_x s_x A_{xi}.$$

For the first term in the right member of this equation we insert its expression taken from (22.6), which gives

$$\sum_k c_k A_{ki}^{(2)} = c_i + \sum_k s_k (e_{ki} + A_{ki}). \tag{22.7}$$

The numbers

$$A_{ki}^{(2)} = \sum_x A_{kx} A_{xi}$$

are the elements of the symbolic square of the matrix A_{ki}.

In the equation (22.7) we again write for a moment x instead of i, postmultiply by A_{xi} and utilise the original equation (22.6). This gives

$$\sum_k c_k A_{ki}^{(3)} = c_i + \sum_k s_k(e_{ki} + A_{ki} + A_{ki}^{(2)}).$$

The numbers

$$A_{ki}^{(3)} = \sum_x A_{kx}^{(2)} A_{xi}$$

are the elements of the symbolic third power of the matrix A_{ki}.

Quite generally we get

$$(22.8)\quad \sum_k c_k A_{ki}^{(N)} = c_i + \sum_k s_k (e_{ki} + A_{ki} + A_{ki}^{(2)} + \cdots + A_{ki}^{(N-1)})$$

where $A_{ki}^{(\nu)}$ are the elements of the ν-th symbolic power of A_{ki}, whose recurrent definition is given by

$$(22.9)\quad A_{ki}^{(\nu+1)} = \sum_x A_{kx}^{(\nu)} A_{xi}$$

with the initial condition $A_{ki}^{(1)} = A_{ki}$.

Let us assume that the numbers $A_{ki}^{(N)}$ tend towards definite limits as N increases. If this is the case, *these limits for any given k must be proportional to the elements in a row of the adjoint of the matrix in (22.5)*. (Since these rows are themselves proportional, it does not matter which row we consider). Indeed, if $A_{ki}^{(\infty)}$ is the limit of $A_{ki}^{(N)}$ as N increases, we must obviously by (22.9) have

$$A_{ki}^{(\infty)} = \sum_x A_{kx}^{(\infty)} A_{xi}.$$

Hence

$$(22.10)\quad \sum_x A_{kx}^{(\infty)} (A_{xi} - e_{xi}) = 0 \qquad (i = 1, 2 \cdots n).$$

Let k be a fixed member and consider the n magnitudes $A_{k1}^{(\infty)}, \cdots, A_{kn}^{(\infty)}$. They are solutions of the homogeneous system (22.10) and must hence be proportional to the elements in a row of the adjoint of the matrix $(A_{xi} - e_{xi})$, q.e.d.

From now on we shall make the assumption that the matrix A_{ki} is such that the sum of the elements in each of its rows is equal to unity, in other words

$$(22.11)\quad A_{k0} = 1.$$

Furthermore, we assume that the sum of the right members is zero, i.e.

$$(22.12)\quad s_0 = 0.$$

If (22.11) and (22.12) are fulfilled, the system (22.5) certainly has a solution. By complete induction, using (22.9), it is easily seen that all the symbolic powers will also have the property (22.11), i.e.

$$(22.13)\quad A_{k0}^{(\nu)} = 1 \text{ for any } \nu \text{ and } k.$$

But then the limit $A_{ki}^{(\infty)}$ must be independent of k. Indeed, the num-

bers $A_{k1}^{(\infty)} \cdots A_{kn}^{(\infty)}$ for two different values of k must be proportional, because both these sets are proportional to the elements in a row of the adjoint of the matrix in (22.5). And by (22.13) the sum of the elements are equal in both sets, namely 1, which is only possible when the elements of the two sets are actually equal. If we let P_i denote the limit towards which $A_{ki}^{(N)}$ tends, we consequently have

$$P_i = \operatorname{Lim} A_{ki}^{(N)} = \text{independent of } k. \tag{22.14}$$

At the limit the left member of (22.8) can consequently be written CP_i where

$$C = \sum_k c_k = c_1 + \cdots + c_n. \tag{22.15}$$

By (22.11) the sum of the P's is unity, i.e.

$$P_0 = 1. \tag{22.16}$$

If $A_{ki}^{(N)}$ tends towards a limit different from zero, it is clear that the series $e_{ki}+A_{ki}+\cdots+A_{ki}^{(N)}$ in the right member of (22.8) must diverge. This need not mean that the whole expression in the right member diverges. Indeed, for high values of N $A_{ki}^{(N)}$ is nearly independent of k, so that the summation is virtually equal to $P_i(s_1+\cdots+s_n)$ which by (22.12) is zero. This fact can be utilized to throw the whole expression into a form involving a convergent series. Since the effect of the higher terms are annihilated by the summation over k, because these higher terms become independent of k, it appears that the feature which really determines the total value of the expression is *how the earlier terms deviate from their limiting value.* This suggests to replace the series in the right member of (22.8) by the series

$$(e_{ki} - A_{ki}^{(N)}) + (A_{ki} - A_{ki}^{(N)}) + \cdots + (A_{ki}^{(N-1)} - A_{ki}^{(N)})$$

which may also be written

$$Q_{ki}^{(N)} = 1.D_{ki}^{(1)} + 2.D_{ki}^{(2)} + \cdots + N.D_{ki}^{(N)} \tag{22.17}$$

where

$$D_{ki}^{(\nu)} = A_{ki}^{(\nu)} - A_{ki}^{(\nu-1)} \qquad D_{ki}^{(1)} = A_{ki} - e_{ki}. \tag{22.18}$$

By (22.13) we see that the row sums are zero both in $D_{ki}^{(\nu)}$ and $Q_{ki}^{(N)}$, i.e.

$$D_{k0}^{(\nu)} = 0 \quad Q_{k0}^{(\nu)} = 0 \text{ for any } \nu \text{ and } k. \tag{22.19}$$

Assuming that $Q_{ki}^{(N)}$ converges we put

$$Q_{ki} = \operatorname*{Lim}_{N \to \infty} Q_{ki}^{(N)} \tag{22.20}$$

so that at the limit the equation (22.8) can finally be written

$$c_i = CP_i + \sum_{k=1}^{n} s_k Q_{ki} \tag{22.21}$$

where P_i is defined by (22.14) and Q_{ki} by (22.20); C is an arbitrary parameter that expresses the sum of the c's.

Under assumption of convergency (22.21) furnishes the solution of any system of equations of the form (22.5) where A_{ki} and s_i satisfy (22.11) and (22.12).

23. Projections in N-Space

Consider an N dimensional space $(x_1 \cdots x_N)$. The symbol x_K may be looked upon either as the point $(x_1 \cdots x_N)$ or as the vector from origine to this point. The individual numbers $(x_1 \cdots x_N)$ we shall call the *total-space* components of the vector, in order to distinguish from certain other kinds of components later to be considered.

Let L be an $N-m$ dimensional linear subspace through origin of the total space. This means that L is the locus of points x_K that satisfy m equations of the form[4]

$$\sum_K f_{iK} x_K = 0 \qquad (i = 1, 2 \cdots m) \tag{23.1}$$

where f_{iK} is an m-rowed and N-columned matrix of rank m. The equations (23.1) express that any vector belonging to L is orthogonal to all the m vectors f_{iK} $(i=1, 2 \cdots m)$.

By the classical rules of linear equations there exists a set of $n=N-m$ linearly independent vectors g_{jK} $(j=1,2 \cdots n)$ belonging to L and such that *any* vector u_K belonging to L can be written as a linear form in the g_{jK}, in other words such that we have

$$u_K = \sum_j \eta_j g_{jK} \qquad (K = 1, 2 \cdots N) \tag{23.2}$$

where the η_j form a set of numbers called the components of u_K in the reference system g_{jK}. The vectors g_{jK} are said to *unfold* L. Since every g_{jK} belongs to L we have the orthogonality relation

$$\sum_K f_{iK} g_{jK} = 0 \quad \begin{pmatrix} i = 1, 2 \cdots m \\ j = 1, 2 \cdots n \\ m + n = N \end{pmatrix}. \tag{23.3}$$

[4] I do not express the following formulae in tensor notation because I shall here make no use of the "up-or-down" character of the affixes, that is of whether the affixes are covariant or contravariant.

In the same way as the g_{jK} unfold L, the f_{iK} unfold a certain linear manifold that is *normal* to L. The vectors g_{jK} may be called the *unfolding* vectors and the f_{iK} the *normality* vectors of L. The matrix g_{jK} may be called the exterior unfolding matrix and f_{jK} the exterior normality matrix L. The expression "exterior" is used to indicate that these matrices depend, not only on the intrinsic properties of the vector bunches g_{jK} and f_{iK} respectively, but also on their position relative to the reference system of the total space.

Consider the square matrices

$$\phi_{ij} = \sum_K f_{iK} f_{jK} \qquad (i, j = 1, 2 \cdots m) \tag{23.4}$$

$$\gamma_{ij} = \sum_K g_{iK} g_{jK} \qquad (i, j = 1, 2 \cdots n) \tag{23.5}$$

ϕ and γ are called the *metric* matrices of L. They are symmetric and may obviously be looked on as moment matrices, hence they are by the Gramian determinant formula positive definite.

Since we assume the matrix $\|f_{iK}\|$ to be of rank m, ϕ_{ij} is non singular, and the same applies to γ_{ij}. We may therefore consider their reciprocals

$$\overset{*}{\phi}_{ij} = \text{the reciprocal of } \phi_{ij} \tag{23.6}$$

$$\overset{*}{\gamma}_{ij} = \text{the reciprocal of } \gamma_{ij} \tag{23.7}$$

By means of the reciprocal metric matrices we form the *reciprocal exterior* matrices

$$\overset{*}{f}_{iK} = \sum_\kappa \overset{*}{\phi}_{i\kappa} f_{\kappa K} \qquad \begin{pmatrix} i = 1, 2 \cdots m \\ K = 1, 2 \cdots N \end{pmatrix} \tag{23.8}$$

$$\overset{*}{g}_{jK} = \sum_\kappa \overset{*}{\gamma}_{j\kappa} g_{\kappa K} \qquad \begin{pmatrix} j = 1, 2 \cdots n \\ K = 1, 2 \cdots N \end{pmatrix}. \tag{23.9}$$

By inverting these equations for a given K we obtain

$$f_{iK} = \sum_\kappa \phi_{i\kappa} \overset{*}{f}_{\kappa K} \tag{23.10}$$

$$g_{jK} = \sum_\kappa \gamma_{j\kappa} \overset{*}{g}_{\kappa K}. \tag{23.11}$$

These relations show that between the exterior matrices and their reciprocals there exist the following relations

$$\sum_K \overset{*}{f}_{iK} f_{jK} = \sum_K f_{iK} \overset{*}{f}_{jK} = e_{ij} \tag{23.12}$$

$$\sum_K \overset{*}{g}_{iK} g_{jK} = \sum_K g_{iK} \overset{*}{g}_{jK} = e_{ij} \tag{23.13}$$

where e_{ij} are the unit numbers $=0$ or 1 accordingly as $i \neq j$ or $i=j$.

Now let us consider the set of $N=m+n$ vectors $f_{1K} \cdots f_{mK}\ g_{1K} \cdots g_{nK}$. They form a set of linearly independent vectors. Indeed the Gramian of this total set is by (23.3).

$$(23.14)\qquad \begin{vmatrix} \phi_{11} \cdots \phi_{1m} & 0 \cdots 0 \\ \cdots\cdots\cdots & \cdots\cdots \\ \phi_{m1} \cdots \phi_{mm} & 0 \cdots 0 \\ 0 \cdots 0 & \gamma_{11} \cdots \gamma_{1n} \\ \cdots\cdots\cdots & \cdots\cdots \\ 0 \cdots 0 & \gamma_{1n} \cdots \gamma_{nn} \end{vmatrix} = |\phi_{ij}| \cdot |\gamma_{ij}| \neq 0$$

Therefore *any* vector x_K in total space can be written in the form

$$(23.15)\qquad x_K = \sum_{i=1}^{m} \xi_i f_{iK} + \sum_{j=1}^{n} \eta_j g_{jkK}.$$

The numbers ξ_i are the components of x_K in the manifold that is normal to L, and the numbers η_j are the components of x_K in L itself.

Between the total space components x_K and the components ξ_i and η_j there exists a unique correspondence. By (23.15) x_K is determined in terms of the ξ_i and η_j. And multiplying (23.15) by $f^*_{\kappa K}$ and $g^*_{\delta K}$ respectively and performing a summation over K we get

$$(23.16)\qquad \xi_i = \sum_K \overset{*}{f}_{iK} x_K \qquad (i = 1, 2 \cdots m)$$

$$(23.17)\qquad \eta_j = \sum_K \overset{*}{g}_{jK} x_K \qquad (j = 1, 2 \cdots n).$$

By means of the exterior matrices and their reciprocals we finally form the *projection* matrices

$$(23.18)\qquad F_{HK} = \sum_i f_{iH} \overset{*}{f}_{iK} = \sum_i \overset{*}{f}_{iH} f_{iK}$$

$$(23.19)\qquad G_{HK} = \sum_j g_{jH} \overset{*}{g}_{jK} = \sum_j \overset{*}{g}_{jH} g_{jK}.$$

The expression for these matrices may also be written

$$(23.20)\qquad F_{HK} = \sum_{ij} \phi_{ij} \overset{*}{f}_{iH} \overset{*}{f}_{jK} = \sum_{ij} \overset{*}{\phi}_{ij} f_{iH} f_{jK}$$

$$(23.21)\qquad G_{HK} = \sum_i \gamma_{ij} \overset{*}{g}_{iH} \overset{*}{g}_{jK} = \sum_i \overset{*}{\gamma}_{ij} g_{iH} g_{jK}$$

Between the two projection matrices there exists the fundamental relation

$$F_{HK} + G_{HK} = e_{HK}. \tag{23.22}$$

To prove this consider the N dimensional linear form $\sum_K F_{HK}x_K$. Inserting here the expression for x_K taken from (23.15) ,and the expression for F_{HK} taken from the second and third member respectively in (23.20) we get

$$\sum_\kappa \xi_\kappa \sum_{ijK} \phi_{ij} \overset{*}{f}_{iH} \overset{*}{f}_{jK} f_{\kappa K} + \sum_\beta \eta_\beta \sum_{ijK}^{*} \phi_{ij} f_{iH} f_{jK} g_{\beta K}.$$

By the summation over K the last term in this expression vanishes on account of (23.3), and the first term reduces to

$$\sum_{ij} \xi_i \phi_{ij} \overset{*}{f}_{jH}.$$

If this expression is multiplied by x_H and summed over H we get by again using (23.15)

$$\sum_{HK} F_{HK}x_H x_K = \sum_{ij} \xi_i \phi_{ij} \sum_H \overset{*}{f}_{jH} \left\{ \sum_\kappa \xi_\kappa f_{\kappa H} + \sum_\beta \eta_\beta g_{\beta H} \right\}.$$

By the summation over H the second term vanishes and the first reduces so that we finally get

$$\sum_{HK} F_{HK}x_H x_K = \sum_{ij} \phi_{ij} \xi_i \xi_j \tag{23.23}$$

similarly we get

$$\sum_{HK} F_{HK}x_H x_K = \sum_{ij} \gamma_{ij} \eta_i \eta_j \tag{23.24}$$

so that

$$\sum_{HK} (F_{HK} + G_{HK}) x_H x_K = \sum_{ij} \phi_{ij} \xi_i \xi_j + \sum_{ij} \gamma_{ij} \eta_i \eta_j. \tag{23.25}$$

This being so, let us consider the N dimensional quadratic form $\sum_{HK} e_{HK} x_H x_K$. Inserting here for x_H and x_K we find easily that the form reduces to the same expression as the one we have in the second member of (23.25). Hence we must have for *any* choice of the x's

$$\sum_{HK} (F_{HK} + G_{HK}) x_H x_K = \sum_{HK} e_{HK} x_H x_K \tag{23.26}$$

which is only possible when (23.22) is fulfilled.

By means of the projection matrices and the fundamental formula (23.22) it is now easy to express the coordinates x_K' of the point obtained by taking an arbitrary point x_K and *projecting it orthogonally to* L. Indeed this projection is defined as the point whose L components are the *same* as the L components of x_K, but whose components in the manifold normal to L *are all zero*. The ξ and η components of the pro-

jection considered are thus immediately given by the very definition of the projection. The thing that interests us is, however, how the total-space coordinates of the projection can be expressed directly in terms of the total-space coordinates x_K. This is obtained by expressing the total-space coordinates of x_K' by means of its ξ and η coordintes using (23.15), then putting here the ξ coordinates equal to zero and the η coordinates equal to those of x_K, and finally by (23.17) to express the η coordinates of x_K by means of its total space coordinates. Doing this we find

$$x_H' = \sum_K G_{HK} x_K. \tag{23.27}$$

By means of the fundamental relation (23.22) this is equivalent to

$$x_H' = \sum_K (e_{HK} - F_{HK}) x_K \tag{23.28}$$

which can also be written

$$x_H' = x_H - \sum_K F_{HK} x_K. \tag{23.29}$$

This explains the name projection matrices given to F_{HK} and G_{HK}.

As an example consider the case $m=1$, that is $n=N-1$. In this case L is simply a plane defined by an equation of the form

$$f_1 x_1 + \cdots + f_N x_N = 0. \tag{23.30}$$

We now get

$$\phi_{11} = f_1^2 + \cdots + f_N^2$$

$$\overset{*}{\phi}_{11} = \frac{1}{\phi_{11}}$$

$$\overset{*}{f}_K = \frac{f_K}{f_1^2 + \cdots + f_N^2}$$

$$F_{HK} = \frac{f_H}{\sqrt{f_1^2 + \cdots + f_N^2}} \cdot \frac{f_K}{\sqrt{f_1^2 + \cdots + f_N^2}}$$

and therefore

$$x_H' = x_H - \frac{f_H}{f_1^2 + \cdots + f_N^2} \sum_K f_K x_K. \tag{23.31}$$

Finally let us determine the projection of x_K, not on L, but on the manifold M that is defined by the *inhomogeneous* equations

$$\sum_K f_{iK} x_K = u_i \qquad (i = 1, 2 \cdots m) \tag{23.32}$$

u_i being a set of given numbers.

The two manifolds L and M are parallel because they have the same normality manifold, namely the one unfolded by the vectors f_{iK} $(i=1,2 \cdots m)$. If x_K'' is the projection of x_K on M and x_K' its projection on L, x_K'' will consequently have the same η coordinates as x_K', that is, as x_K itself. And the ξ coordinates of x_K'' must obviously be just large enough to satisfy the equations (23.22). In other words we must have

$$\sum_K f_{iK}\left\{\sum_\kappa \xi_\kappa'' f_{\kappa K} + \sum_j \eta_j g_{jK}\right\} = u_i$$

where ξ_κ'' are the ξ components of x_K''. The summation over K in the last of the above terms give 0, so that the equation reduces to

$$\sum_j \phi_{ij}\xi_j'' = u_i \tag{23.33}$$

hence

$$\xi_i'' = \sum_j^* \phi_{ij} u_j. \tag{23.34}$$

In the expression for the total-space components of x_K'', namely

$$x_H'' = \sum_i \xi_i'' f_{iH} + \sum_j \eta_j g_{jH}$$

we notice that the last term just gives the previously calculated coordinate x_K', i.e. the expression (23.28). And the first term in the above expression reduces by (23.34) to

$$\sum_{ij} u_i \overset{*}{\phi}_{ij} f_{jH} = \sum_i u_i \overset{*}{f}_{iH}.$$

Hence: *If we take an arbitrary point x_K and project it on to the linear manifold* (23.32) *we get a point x_K'' whose total-space coordinates are*

$$x_H'' = \sum_{i=1}^{m} u_i \overset{*}{f}_{iH} + \sum_{K=1}^{N} (e_{HK} - F_{HK}) x_K \tag{23.35}$$

where $\overset{*}{f}_{iH}$ is the reciprocal exterior matrix and F_{HK} the projection matrix defined above.

In the special case $m=1$ we get

$$x_H'' = x_H + \frac{f_H}{f_1^2 + \cdots + f_N^2}\left(u - \sum_K f_K x_K\right) \tag{23.36}$$

where u is the constant in the second member of the equation defining the plane on to which the projection is made.

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THE MEETING OF THE ECONOMETRIC SOCIETY IN BERKELEY, CALIFORNIA, JUNE, 1934

THE thirteenth meeting of the Econometric Society was held in Berkeley, California, from Wednesday afternoon June 20, 1934, through Friday evening, June 22. All sessions were held jointly with Section K, A.A.A.S. Section A of the Association, the American Mathematical Society, and the American Statistical Association, joined in the first session. The latter society also joined in the sessions of Thursday afternoon and of Friday afternoon and evening. The program was arranged by John B. Canning and Holbrook Working of Stanford University.

At the Wednesday afternoon session four papers were read and discussed. E. B. Wilson of Harvard University presented a simplified, complete solution of Boole's "Challenge Problem"—an indeterminate problem in probabilities in two independent variables. He pointed out the usefulness of the solution in obtaining the limiting values of probabilities in analogous problems in statistical practice in which, for want of completeness of classification in the original tally, the required probability of an event remains indeterminate. The solution was obtained, in part, by means of Yule's fourfold association tables.

In his comment on this paper James V. Uspensky of Stanford University suggested that the solution might also be obtained by identifying and evaluating the fundamental inequalities in the system of simultaneous inequalities presented by the problem. This method, graphically expressed, leads to the bounding of a polygon all points within which satisfy the conditions of the problem.

Conditions which demand and supply functions for groups of related commodities may be expected to satisfy were derived by Harold Hotelling of Columbia University in the *Journal of Political Economy* for October, 1933, on the assumption that the buyers were entrepreneurs, or others buying in such a way that the budget could be increased without effective limit in case the purchases should lead to sufficiently increased profits. Both equations and inequalities to be satisfied by the first derivatives of the demand functions were obtained. In the present paper he generalized these equations and inequalities to the case of a buyer with a strictly limited budget. The inequalities are obtained from conditions on the second derivatives of a function of n variables subject to restraint, which function is to be a maximum. Taking the demand functions in the form $p_i = f_i(q_1, \cdots, q_n)$, and putting $p_{ij} = \delta p_i / \delta q_j$, the inequalities consist in the requirement that each of the determinants of the form

$$\begin{vmatrix} p_{11} & p_{12} & \cdots & p_{1m} & p_1 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ p_{m1} & p_{m2} & \cdots & p_{mm} & p_m \\ p_1 & p_2 & \cdots & p_m & 0 \end{vmatrix}$$

should have the sign $(-)^m$; this for every set of m commodities, and for every value of m from 1 to n, the total number of commodities. These are conditions upon the demand functions for individuals. For the total demand they may not hold, but will hold if the symmetrical determinants of the p_{ij} are all positive.

He pointed out that mathematical investigations of demand for commodities has, for some time, been on the verge of becoming practically useful. Each additional realistic condition imposed, therefore, helps to develop this branch of mathematical economics into a working tool for realistic problems.

John M. Thompson of the Rice Institute presented a mathematical formulation of the theory of production stages found in the writings of F. A. Hayek and specified conditions under which Hayek's conclusions would be valid. In the floor discussion of this paper it was variously suggested that the conditions assumed were inadequate to represent economic realities and that certain significant economic variables were omitted. To these comments John B. Canning replied that the supposedly omitted economic variables would be found, on close study, to be adequately accounted for in the simpler system of variables defined by Thompson. E. B. Wilson, supplementing Canning's remark, showed the necessity, in mathematical formulation, for sets of elementary variables that, by their definitions, are mutually exclusive. Fritz Machlup of Vienna, Austria, upheld Thompson's statement that the postulated conditions conformed to Hayek's expressed and implied conditions.

Dunham Jackson of the University of Minnesota demonstrated that certain portions of the theory of small samples, hitherto dealt with in the standard memoirs by advanced mathematical treatment, are capable of straightforward demonstration by much simpler methods. He illustrated the characteristic operations in calculation, for distributions of arbitrary form, such secondary frequency functions as those of the standard deviation of a mean and of the mean of the squares of the standard deviations of samples drawn from the assumed primary distribution. He also treated the evaluation of integrals of these secondary functions. In the calculation of distributions of standard deviations and of Student's z for samples from a normal population, he

showed that n-dimensional geometry can be replaced by analysis based on linear orthogonal transformations of variables.

In the first paper of the Thursday morning session Raymond Garver of the University of California at Los Angeles generalized in a single equation a graphical discussion of Yntema's[1] on the effect of dumping on monopoly price. Given:

1. A monopolist operating in a domestic market enters a foreign market (in which he is also a monopolist);

2. The demand functions for the monopolist's product in the two markets are linear; and

3. The monopolist's cost of producing x units per unit of time is expressed by Ax^2+Bx+C.

Required: The effect, on the amount offered and the price charged in the domestic market, of the monopolist's entering the foreign market.

This effect is given by $x_a=x_m-Ax_b/k$ wherein k is a positive constant, x_m is the optimum production *before* opening the second market, x_a is the optimum amount to be offered in the domestic market *after* opening the second market, and x_b is the optimum amount to be offered in the foreign market. It will be noted at once that the results depend upon the sign of A: if A is negative—that is, if operations are subject to decreasing marginal cost—the amount sold in the domestic market will be increased and the price lowered; if A is positive, the opposite effect upon domestic price and supply are indicated.

When more general demand and cost functions are considered, the controlling factor still seems to be whether the monopolist is subject to conditions of increasing or of decreasing marginal cost.

J. M. Keynes' *Treatise on Money*, considered as a proposal for policy in the economic arts (in contradistinction to a contribution to economic science), was ably discussed by Leo Rogin of the University of California. As a proposed means of minimizing the amplitude of short-run oscillations of business activity in our course of shifting from one equilibrium level to another, it is not to be judged on the basis of probable success alone. If to obtain the benefits of the proposal it becomes necessary to alter established institutions (or to introduce new ones) about which there are ethical preconceptions, these subjective judgments must be taken into account also.

Lewis A. Maverick of the University of California at Los Angeles contributed a statistical and graphical study of the history of real estate activity in certain California counties. This paper will shortly appear in the *Journal of Land Economics and Public Utilities*; it is one of a series of which two earlier sections have been published in that organ.

[1] *Journal of Political Economy*, XXXVI, 686.

These studies strongly reflect the repercussions of changes in national and world economic conditions upon localized activities.

The Thursday afternoon meeting was devoted to price studies and to methods of price analysis. H. E. Erdman of the University of California reported an extraordinary degree of success, on the part of the California Walnut Grower's Association, in naming an opening season price which, with only minor price changes and/or adjustments in selling program, will market the whole crop in the holiday trade season.

From these prices, adjusted for changes in the general price level, for population increases, and for proportion of high grade nuts to total yield, he has obtained a very tight fit to a plausibly smoothed supply-price curve. He pointed out, however, that even for such a specialty commodity the number of seasons' experience is too short to make the fit reliable. He assigned partial reasons for supposing that somewhat lower prices might have increased the net returns to growers.

Certain of the difficulties encountered in attempts to obtain supply-price relations statistically were enumerated and discussed by H. J. Stover of the University of California. He expressed doubt that, aside from specialties in the luxury class, satisfactory evaluations will be found. Supplementing Stover's paper, Merrill K. Bennett of Stanford University reported that after many years of continued economic study of wheat markets at the Food Research Institute no satisfactory basis of determining supply-price relations had been discovered.

Holbrook Working of Stanford University discussed "Differential Price Behavior as a Subject for Commodity Price Analysis." Price behavior that defies analysis by the methods of multiple correlation may often be investigated with highly satisfactory results by the study of price differences and of ratios of price differences to the concurrent terms in one of the price series. The ideal materials consist of two series, one of which responds exactly like the second to all influences controlling the second and varies, in addition, with changes in some simple group of special influences. But any two series will serve provided that the difference series be dominated by influences that are obscured in the minuend and subrahend series by more potent common components. He referred to several major research monographs in which this method has been a principal instrument. He illustrated by means of wall charts the application of the method to the study of spreads among wheat futures and between cash wheat prices and the several futures prices.

In conclusion he remarked that economists familiar with wheat markets and professional wheat traders might, on *a priori* grounds, concur in expecting some of the results that laborious statistical analysis produces; but probably there would also be a consensus in expectations

not in accord with statistical facts; some cleavages and clashes of expectations would be likely also. The special merit of conclusions based on statistical analysis—as compared to those founded in *a priori* hypotheses and unsystematic fact study—lies not in their form and content but in their superior trustworthiness.

Robert D. Calkins of the University of California, by way of illustrating the superior order of evidence furnished by this method, enumerated several beliefs long commonly held by wheat traders and by economists for which, a forthcoming study of his show, there is no foundation in fact. These beliefs include the following items: (1) that there is a fundamental bullish tendency among traders in export markets and a corresponding bearish tendency among traders in import markets; (2) that when North American wheat prices are above the export to Liverpool point they take on the character of domestic prices, i.e., cease to move in close unison with Liverpool prices; and (3) that wheat price differences between export points and import points tend to equal the costs of transfer. The facts show that the characteristic difference is significantly less than the costs of transfer.

Working's paper and Calkins' comment upon it carry the hall-mark common to that aristocracy who write on methods only *after* they have proved the fruitfulness of their techniques in actual research studies.

The Friday morning session was devoted to three papers in economic theory. Lassere's scheme of classification of enterprises was sketched and commented upon by G. H. Bousquet of the Université d'Alger. He urged the merits of this scheme, the basis of which consists of differences in modes of distributing the sales proceeds of goods (and/or services) among those economic classes (or factors) that contribute to the production of the goods sold.

In commenting on the paper, B. F. Haley of Stanford University expressed doubt that this or any other formalized general scheme of classification is at all likely to be more fruitful than partial and special purpose classifications adopted for their convenience in special research studies. He pointed out that particular enterprises, unlike particular organisms in the Linnean biological scheme, must be expected to shift from class to class in an erratic manner.

After neatly sketching the character of differential cost analysis that should be made in choosing between alternative economic choices in enterprise operations, E. L. Grant of Stanford University illustrated, both by specific real cases from his engineering experience and by references to accepted theory writings in economics and in engineering, the major errors of cost estimating. These characteristically take one of the following forms: (1) an illicit substitution of total (or average) cost, past and future, for the total (or average) differential cost special

to the choice made; (2) illicit substitution of total (or mean) future outlays for the summation (or mean) of future outlays contingent upon the particular option chosen; and (3), in those cases which are recognized as increment cost cases, the seriously incomplete summation of the increments to future outlays contingent upon the option elected. The literatures dealing with optimum lot sizes in manufacturing, with the replacement of major structures and equipment in industry, and with the economies of reduced highway distances—as well as practice in these fields—abound with false ideas of economy. Both cost accounting and capital and income accounting procedures fail to provide suitable data for sound estimating.

H. R. Hatfield of the University of California remarked that, in view of the extant sound theoretical writing on the subject both in economics, accountancy and engineering, the deficiencies were mostly to be found in practice. He also pointed out that in certain cases the appropriate mean increment cost differs negligibly from mean total cost.

Fritz Machlup, of Vienna, Austria, in a closely reasoned and aptly illustrated paper, took issue with the commonly accepted view—often the basis of state economic policies—that lowered interest rates encourage and enhance production chiefly through lowered production costs. He showed analytically that the effect of lowered rates on unit costs must always be negligibly small as compared with the effect on new fixed investment by way of the enhanced capitalization of future returns. Unless the artificially lowered rates become negative and arithmetically large they cannot reduce unit costs to anything like the degree possible by means of apparently small changes in unit labor costs, unit materials costs, or unit selling prices.

John B. Canning, in commenting on this paper remarked that though the concept of interest as a cost factor contains much in common with the concept of interest as a capitalization factor, they are not identical notions. He pointed out that small reductions of rates on long period loans may, by expediting new fixed investment in young industries, be more effective, as recovery measures, than negative rates on short term loans to established and fully equipped concerns in old industries. He expressed the opinion that Machlup had made a notable contribution to interest theory.

At the Friday afternoon session Norman J. Silberling of San Francisco, after a detailed examination of the concepts represented by the symbols in Fisher's equation of exchange, considered the influence on general price movements of commodity surpluses. He criticized American writings on monetary theory, especially Fisher's, for their too exclusive regard for monetary factors. Too little attention is given, he

suggested, to failures to adjust production rates to consumption (and utilization) requirements. The accumulation of commodity surpluses in the face of uncontrolled production rates may greatly impair estimates of an economic situation drawn from statistical evaluation, of the equation of exchange.

J. S. Davis of Stanford University, in discussing this paper, outlined the critical situation in the world wheat market that arose from the huge carry-over of the 1928 crop and the failure, despite a smaller 1929 crop, to reduce the surplus stock. He discussed the repercussions of wheat price behavior in these years on other prices. He enumerated several other important commodities in which concurrent surpluses had developed, with like consequences upon prices.

Proofs of two theorems involving Foster's generalized mean value function, $f(x)=\left(\frac{\sum_{i=1}^{n}R_i{}^x}{n}\right)^{1/x}$ were offered by John B. Canning of Stanford University as special illustrations of study of a long known, important, but neglected type of frequency function. The theorems presuppose that the graph of the frequency distribution of log R_i is symmetrical with respect to the normal to the log R axis through the log of the geometric mean of the R's.

Theorem 1: $f(x)\cdot f(-x)=\overline{f(0)}^2$; and

Theorem 2: For $x>0$, $f(x)$ is an increasing function of dispersion of the R's; and $f(-x)$ is a decreasing function of the R's. $f(0)$ is independent of changes in dispersion.[2]

Corollary 1: $f'(x)$ is a generalized mean deviation function.

Corollary 2: $f(x)\cdot f(-x)/\overline{f(0)}^2$ is a generalized measure of asymmetry if the symmetry condition of the theorems is a statistical tendency rather than an absolute condition.

The study of corresponding relations of the averages for other types of distribution, is of prime importance in index numbers; for, in that field, the dispersion of the frequency distributions of price relatives increases as the date of comparison recedes from the base date.

In his comment on the paper, Mr. Nilan Norris of Stanford University attributed the unsatisfactory progress in index numbers to the gap in statistical theory rather than to want of factual knowledge of the characteristics of frequency distributions of price relatives. He

[2] Others have proved of this function (though the proofs are not all published): that $\lim_{x\to 0}f(x)$ = the geometric mean of the R's; that $\lim_{x\to\infty}f(x)$ = the largest R; that $\lim_{x\to-\infty}f(x)$ = the smallest R; and that $f'(x)$ is continuous and positive for all real x. For $x=-1$, 1, and 2, $f(x)$ becomes, successively, the harmonic, arithmetic and quadratic mean. But $f(x)$ is an average for all real values of the argument.

emphasized that skewness of distributions is but a special case of asymmetry.

Gerhard Tintner of the Konjunkturforschungs Institut of Vienna contributed a noteworthy discussion of the analysis of time series. Restricting the mathematical assumptions to those which possess economic plausibility, he first discussed the use of Okar Anderson's "variate difference" method of eliminating casual or episodic changes from those that are inherent in the "mathematical expectation." Beginning with $w_i = m_i + x_i$—wherein w_i is the general term in the observed series, x_i the casual component, and m_i the stable economic time function—successive finite differentiation leads to $\Delta^k w_i = \Delta^k m_i + \Delta^k x_i$ in which, as k increases, $|\Delta^k m_i|$ tends to become negligibly small and $\Delta^k x_i$ tends to remain constant.

He next discussed the use of Shepphard's smoothing formula in obtaining the series as it might have been expected to develop in the absence of casual influences. Assuming the function thus obtained to be continuous and differentiable and that it can be written $m = f(s, k, t)$ wherein s represents a seasonal variable, k represents a cyclical variable, and t a non-periodic trend, the function can be expanded in a Taylor's series $m = f(s, k, t) = f_0 + f_s \cdot s + f_k \cdot k + f_t \cdot t + \frac{1}{2} f_{ss} \cdot s^2 + \cdots$ in which f_0 is an initial value free of seasonal, cyclical, and trend influences, f_s, f_k, and f_t are partial derivatives of f with respect to the subscript variables, $f_{ss}, \cdots$ are the like partial derivatives of the second order, etc. In those cases in which the terms involving second and higher order partial derivatives can be neglected, the method of separation is that of the well known technique of moving averages.

Holbrook Working remarked that Tintner's paper aptly showed the characteristic difference between the modes of procedure of American and European students of the subject. The latter are urgently concerned with the development of general methods, the former with devising analyses suited to their immediate special objectives and to the particular series under consideration. No *Methodenstreit* is likely to develop, since both schools are hard-minded enough to recognize the shortcomings and merits of both modes. The variate difference method, he said, works satisfactorily sometimes; for other series either the m_i component may be insignificant or episodic events may have lasting consequences. The erratic character of seasonal movements—both with respect to time of beginning, duration and amplitude of fluctuation—or the unsatisfactorily marked period of the business cycle, or the spurious character of the apparent trend, make the use of this method unsatisfactory in other cases.

At the dinner meeting held at the Bellevue Hotel in San Francisco on Friday evening the results of two major pieces of research were

presented. W. F. C. Nelson of the Cowles Commission reported and discussed the Commission's new index number of American industrial and public utility common stocks. The index is completed for the interval of 1872–1906; shortly it will be extended to 1917; from that date forward the Standard Statistics Company's weekly index numbers are satisfactory. Nelson's own summary is contained in the following paragraph.

Despite the monetary magnitude of stock trading, the sensitivity of stock prices to changing economic situations, the wide popular repercussions of stock price changes, and the mass of statistical information available, the published work on stock prices does not compare in merit and adequacy with what has been done on commodity prices. The Cowles Commission surveyed the 326 common stock price indexes that have been published and found no long-term series that included the features of proper corrections, adequate sampling, and proper weighting and averaging. To remedy this condition the Commission has prepared new indexes, now completed for the interval 1872–1906, incorporating these features. The indexes embody corrections for all rights, splits, and capital changes. They embrace all stocks quoted on the New York Stock Exchange for two successive months, and weight every issue according to shares outstanding. Comparison of the Cowles Commission Indexes with those of the Federal Reserve Bank of New York and Dow Jones Industrial Averages from 1872–1906 revealed general agreement as to direction, but significant differences as to the violence of movements and cyclical peaks and troughs. Comparison with the Axe-Houghton Index from 1883–1906 revealed large discrepancies as to direction of movements, amplitude of fluctuations, and cyclical turning points. The Cowles Commission Indexes embrace samples sometimes six times as large as do these other series and compel some revision of ideas of stock price action founded on earlier indexes. An index of yields properly weighted by the dollar size of dividends paid was also presented.

Samuel Moment of Stanford University, who led the discussion on the paper, said, in part, that the preparation of an index number of stock prices which is not open to serious objections will serve chiefly to fix attention upon our ignorance of investment experience. Despite the wealth of reliable factual data we know little or nothing of the investment experience either of particular classes of traders, of customers of particular classes of brokers, or of investors in particular classes of securities. We know little of the consequences to the public of pool operations and other special practices. He cited the recent Senate investigations and certain recent partial studies of lists of speculative accounts which show that security market regulation, if it is to be

effective in the public interest, must be based on studies still to be made.

W. L. Crum of Harvard University made a partial report on a study of the correlation between several operating ratios, on the one hand, and mean investment in assets, on the other, of American business corporations.[3] The ratios considered in the paper were: (1) the ratio of net income to total receipts; (2) the ratio of net income to net worth; and (3) the ratio of current assets to current liabilities. The size classification of concerns is that given in the Treasury's *Statistics of Income*, 1931, for the reporting corporations.

Subclassifications by type of industry and by type of product were separately considered. Corporations which reported a profit were considered separately from those which did not. Subject to reservations by reason of the abnormal character of the single year for which size groups are available and by reason of differences of tendency for concerns to be large (or small) in certain types of industry and in certain sorts of product, there appears to be a positive correlation of investment in assets with each of the three ratios referred to. There is some internal evidence that in prosperous years small corporations may, as a class, show higher rates of return on net worth than large concerns. A slight tendency for the ratios to be most favorable to the size-class next below the largest concerns, suggests that the regressions may be curvilinear but the data are insufficient to establish this.

T. J. Kreps of Stanford University suggested various phenomena—some special to the year in question, others peculiar to differences in capital structure and to differences in the age distribution of the several classes—which may, when data become available, help to resolve some of the doubts now felt about the trustworthiness of the present indicated correlations.

In addition to the Society's own program, the members found much to interest them in the Conference on Land-Use Planning sponsored jointly by Sections K and O, in two sessions of Section K on monetary theory and problems of consumption,[4] in the general evening address of E. B. Wilson "Are There Periods in American Business Activity,"[5] and in the Hector Maiben Lecture on "Planning the Land of the Future" by Dr. L. D. Stamp of the London School of Economics.

JOHN B. CANNING

Stanford University,

[3] The study has since been published in full in a Research Bulletin of the Harvard Graduate School of Business Administration.

[4] Notes on the papers read at these sessions are published in *Science*, July 20, 1934.

[5] Based on his paper in the *Quarterly Journal of Economics* of May, 1934.

OBITUARY

ECONOMETRICA records with deep regret that the deaths of the following members have taken place in the four years that have elapsed since the founding of the Econometric Society.

Professor Thomas S. Adams
New Haven
Connecticut

Professor E. B. Bouvier
Université de Lyon
Lyon, France

Professor Henry W. Farnam
New Haven
Connecticut

Mr. Joseph Kitchin
London
England

Ass't Professor Stan. Kohn
School of Laws
Praha, Czechoslovakia

M. Lucien March
Ancien directeur de la Statistique
générale de la France
Paris, France

Mr. J. Edward Meeker
New York Stock Exchange
New York

Dr. Josef Mráz
Vice-President of the State
Statistical Office
Praha, Czechoslovakia

PAPERS TO APPEAR IN ECONOMETRICA

THE SURVEYS

J. R. Hicks: Survey of General Economic Theory (to appear January 1935)

J. Tinbergen: Survey of Business Cycle Theory (to appear April 1935)

Erich Schneider: Survey of General Economic Theory (to appear January 1936)

OTHER CONTRIBUTIONS TO APPEAR IN EARLY ISSUES

Léon Walras et sa Corréspondance avec Augustin Cournot et Stanley Jevons, avec une note d'Étienne Antonelli

Henry A. Babcock: Maximum Value of Urban Land Converted to Divers Uses

Ragnar Frisch and Harald Holme: The Characteristic Solutions of a Mixed Difference and Differential Equation Occurring in Economic Dynamics

C. H. P. Gifford: The Period of Production Under Continuous Input and Point Output in an Unprogressive Community

Benjamin Greenstein: Periodogram Analysis with Special Application to Business Failures

Harold Hotelling: Demand Functions with Limited Budgets

M. Kalecki: A Macrodynamic Theory of Business Cycles

C. R. Noyes: Stable Prices *vs* Stable Exchanges

Erich Schneider: Bemerkungen zu einer Theorie der Raumwirtschaft

Heinrich Baron von Stackelberg: Vilfredo Pareto

Edward Theiss: Dynamics of Saving and Investment

S. S. Wilks: On the Independence of Sets of Normally Distributed Statistical Variables

GROWTH AND DISTRIBUTION OF THE MEMBERSHIP OF THE ECONOMETRIC SOCIETY

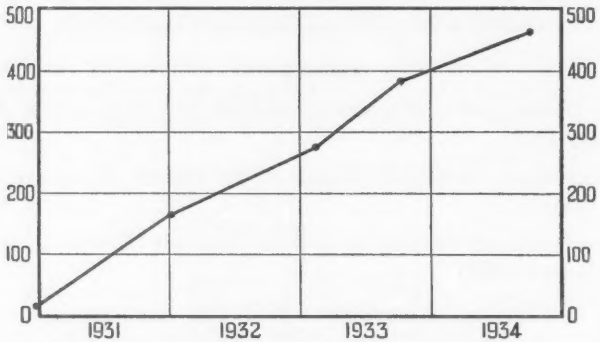

FIGURE 1.—Number of Members of the Econometric Society as of December 29, 1930, January 1, 1932 (these 163 constitute Charter Members), February 11, 1933, October 6, 1933, and October 1, 1934.

GEOGRAPHICAL DISTRIBUTION OF ECONOMETRIC SOCIETY MEMBERS

United States	151	Denmark	11	Union of South	
Italy	42	Czechoslovakia	10	Africa	3
France	41	Switzerland	9	Egypt	3
England	28	India	6	Spain	3
Germany	23	Japan	6	Chile	3
Holland	22	Roumania	5	Philippine Is.	3
Poland	15	Argentina	5	Australia	2
Hungary	14	Norway	5	Canada	2
Austria	13	Yugoslavia	4	Finland	1
China	12	Belgium	4	Algeria	1
Sweden	12	Bulgaria	3	Ireland	1
				Total	463

NUMBER OF ECONOMETRIC SOCIETY MEMBERS PER ONE MILLION OF POPULATION, BY COUNTRIES

Denmark	3.08	Chile	.69	Yugoslavia	.29
Holland	2.74	Czechoslovakia	.68	Roumania	.28
Switzerland	2.21	Belgium	.49	Philippine Is.	.24
Sweden	1.95	Bulgaria	.49	Egypt	.20
Austria	1.93	Poland	.47	Canada	.19
Norway	1.78	Argentina	.43	Algeria	.15
Hungary	1.60	U. of South Africa	.37	Spain	.13
United States	1.22	Germany	.36	Japan	.09
Italy	1.02	Ireland	.34	China	.03
France	.98	Australia	.31	India	.02
England	.70	Finland	.30		

LIST OF MEMBERS OF THE ECONOMETRIC SOCIETY

As of October 1, 1934.

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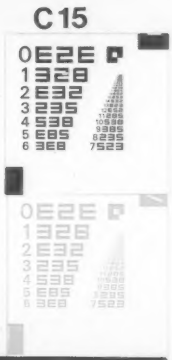
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